Arkansas School of Mathematics, Sciences and the Arts

AP Calculus Free Responses
Categorized by Topic

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1992 AB3

Let f be the function given by $f(x) = \ln\left|\frac{x}{1+x^2}\right|$.

(a) Find the domain of f.

(b) Determine whether f is an even function, an odd function, or neither. Justify your conclusion.

(c) At what values of x does f have a relative maximum or a relative minimum? For each such x, use the first derivative test to determine whether $f(x)$ is a relative maximum or a relative minimum.

(d) Find the range of f.

1989 BC6

Let f be a function that is everywhere differentiable and that has the following properties.

(i) $f(x+h) = \dfrac{f(x)+f(h)}{f(-x)+f(-h)}$ for all real numbers h and x.

(ii) $f(x) > 0$ for all real numbers x.

(iii) $f'(0) = -1$.

(a) Find the value of $f(0)$.

(b) Show that $f(-x) = \dfrac{1}{f(x)}$ for all real numbers x.

(c) Using part (b), show that $f(x+h) = f(x)f(h)$ for all real numbers h and x.

(d) Use the definition of the derivative to find $f'(x)$ in terms of $f(x)$.

1989 AB4

Let f be the function given by $f(x)=\dfrac{x}{\sqrt{x^2-4}}$.

(a) Find the domain of f.

(b) Write an equation for each vertical asymptote to the graph of f.

(c) Write an equation for each horizontal asymptote to the graph of f.

(d) Find $f'(x)$.

1990 AB2

Let f be the function given by $f(x) = \ln \dfrac{x}{x-1}$.

(a) What is the domain of f?

(b) Find the value of the derivative of f at $x = -1$.

(c) Write an expression for $f^{-1}(x)$, where f^{-1} denotes the inverse function of f.

1990 AB6

Let f be the function that is given by $f(x) = \dfrac{ax+b}{x^2 - c}$ and that has the following properties.

(i) The graph of f is symmetric with respect to the y-axis.

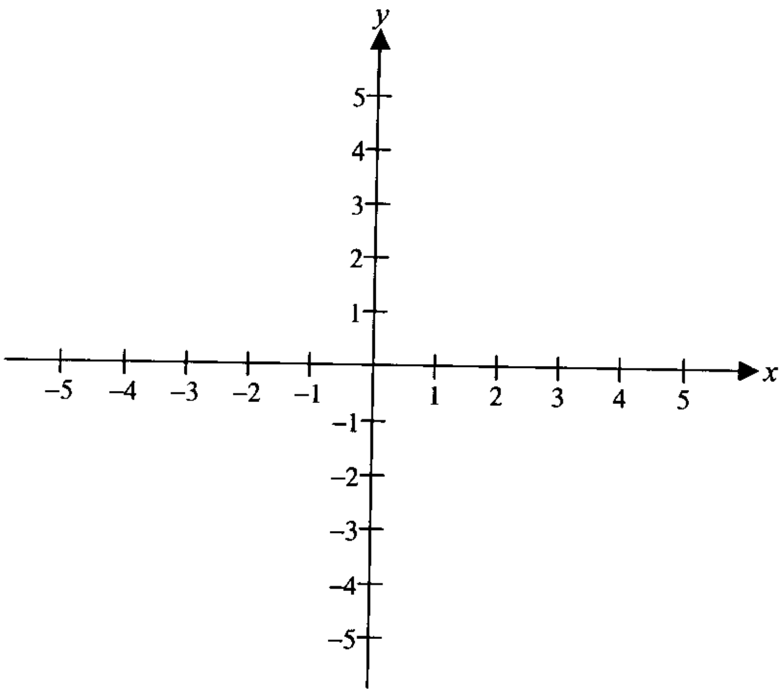
(ii) $\lim_{x \to 2^+} f(x) = +\infty$

(iii) $f'(1) = -2$

(a) Determine the values of a, b, and c.

(b) Write an equation for each vertical and each horizontal asymptote of the graph of f.

(c) Sketch the graph of f in the xy-plane provided below.



1991 AB4

Let f be the function given by $f(x) = \dfrac{|x| - 2}{x - 2}$.

(a) Find all the zeros of f.

(b) Find $f'(1)$.

(c) Find $f'(-1)$.

(d) Find the range of f.

1994 BC 6

Let f and g be functions that are differentiable for all real numbers x and that have the following properties.

(i) $f'(x) = f(x) - g(x)$

(ii) $g'(x) = g(x) - f(x)$

(iii) $f(0) = 5$

(iv) $g(0) = 1$

(a) Prove that $f(x) + g(x) = 6$ for all x.

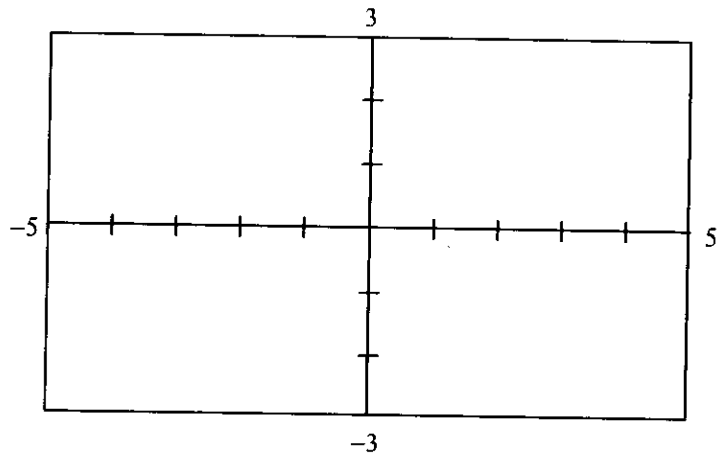
(b) Find $f(x)$ and $g(x)$. Show your work.

1995 AB1

Let f be the function given by $f(x) = \dfrac{2x}{\sqrt{x^2+x+1}}$.

(a) Find the domain of f. Justify your answer.

(b) In the viewing window provided below, sketch the graph of f.



[Viewing Window]
$[-5,5]\times[-3,3]$

(c) Write an equation for each horizontal asymptote of the graph of f.

(d) Find the range of f. Use $f'(x)$ to justify your answer.

Note: $f'(x) = \dfrac{x+2}{(x^2+x+1)^{\frac{3}{2}}}$

1998 Calculus BC Scoring Guidelines

2. Let f be the function given by $f(x) = 2xe^{2x}$.

   (a) Find $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to \infty} f(x)$.

   (b) Find the absolute minimum value of f. Justify that your answer is an absolute minimum.

   (c) What is the range of f?

   (d) Consider the family of functions defined by $y = bxe^{bx}$, where b is a nonzero constant. Show that the absolute minimum value of bxe^{bx} is the same for all nonzero values of b.

(a) $\lim\limits_{x \to -\infty} 2xe^{2x} = 0$

$\lim\limits_{x \to \infty} 2xe^{2x} = \infty$ or DNE

$2 \begin{cases} 1\text{: } 0 \text{ as } x \to -\infty \\ 1\text{: } \infty \text{ or DNE as } x \to \infty \end{cases}$

(b) $f'(x) = 2e^{2x} + 2x \cdot 2 \cdot e^{2x} = 2e^{2x}(1 + 2x) = 0$

if $x = -1/2$

$f(-1/2) = -1/e$ or -0.368 or -0.367

$-1/e$ is an absolute minimum value because:

(i) $f'(x) < 0$ for all $x < -1/2$ and

$f'(x) > 0$ for all $x > -1/2$

-or-

(ii) $f'(x)$ $-$ $+$ at $-1/2$

and $x = -1/2$ is the only critical number

3
- 1: solves $f'(x) = 0$
- 1: evaluates f at student's critical point
  - 0/1 if not local minimum from student's derivative
- 1: justifies absolute minimum value
  - 0/1 for a local argument
  - 0/1 without explicit symbolic derivative

Note: 0/3 if no absolute minimum based on student's derivative

(c) Range of $f = [-1/e, \infty)$

or $[-0.367, \infty)$

or $[-0.368, \infty)$

1: answer

Note: must include the left-hand endpoint; exclude the right-hand "endpoint"

(d) $y' = be^{bx} + b^2xe^{bx} = be^{bx}(1 + bx) = 0$

if $x = -1/b$

At $x = -1/b$, $y = -1/e$

y has an absolute minimum value of $-1/e$ for all nonzero b

3
- 1: sets $y' = be^{bx}(1 + bx) = 0$
- 1: solves student's $y' = 0$
- 1: evaluates y at a critical number and gets a value independent of b

Note: 0/3 if only considering specific values of b

Question 5

The derivative of a function f is given by $f'(x) = (x-3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$

Therefore, f has a relative minimum at $x = 3$.

$2: \begin{cases} 1: \text{minimum at } x = 3 \\ 1: \text{justification} \end{cases}$

(b) $f''(x) = e^x + (x-3)e^x = (x-2)e^x$

$f''(x) > 0$ for $x > 2$

$f'(x) < 0$ for $0 < x < 3$

Therefore, the graph of f is both decreasing and concave up on the interval $2 < x < 3$.

$3: \begin{cases} 2: f''(x) \\ 1: \text{answer with reason} \end{cases}$

(c) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 7 + \int_1^3 (x-3)e^x\,dx$

$$u = x - 3 \quad dv = e^x\,dx$$
$$du = dx \quad v = e^x$$

$$f(3) = 7 + (x-3)e^x\Big|_1^3 - \int_1^3 e^x\,dx$$
$$= 7 + \left((x-3)e^x - e^x\right)\Big|_1^3$$
$$= 7 + 3e - e^3$$

$4: \begin{cases} 1: \text{uses initial condition} \\ 2: \text{integration by parts} \\ 1: \text{answer} \end{cases}$

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Question 4

Let f be the function defined for $x > 0$, with $f(e) = 2$ and f', the first derivative of f, given by $f'(x) = x^2 \ln x$.

(a) Write an equation for the line tangent to the graph of f at the point $(e, 2)$.

(b) Is the graph of f concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

(c) Use antidifferentiation to find $f(x)$.

(a) $f'(e) = e^2$

An equation for the line tangent to the graph of f at the point $(e, 2)$ is $y - 2 = e^2(x - e)$.

$2 : \begin{cases} 1 : f'(e) \\ 1 : \text{equation of tangent line} \end{cases}$

(b) $f''(x) = x + 2x \ln x$.

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of f is concave up on $(1, 3)$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) Since $f(x) = \int \left(x^2 \ln x\right) dx$, we consider integration by parts.

$$u = \ln x \qquad dv = x^2\, dx$$
$$du = \frac{1}{x}\, dx \qquad v = \int \left(x^2\right) dx = \frac{1}{3}x^3$$

Therefore,

$$\begin{aligned} f(x) &= \int \left(x^2 \ln x\right) dx \\ &= \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3 \cdot \frac{1}{x}\right) dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C. \end{aligned}$$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3} \ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$4 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{uses } f(e) = 2 \\ 1 : \text{answer} \end{cases}$

Question 4

Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of h concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of h at $x = 4$.

(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

(a) $h'(x) = 0$ at $x = \pm\sqrt{2}$

$h'(x)$	$-$	0	$+$	und	$-$	0	$+$
x		$-\sqrt{2}$		0		$\sqrt{2}$	

Local minima at $x = -\sqrt{2}$ and at $x = \sqrt{2}$

$$4 : \begin{cases} 1 : x = \pm\sqrt{2} \\ 1 : \text{analysis} \\ 2 : \text{conclusions} \\ \quad < -1 > \text{ not dealing with discontinuity at } 0 \end{cases}$$

(b) $h''(x) = 1 + \dfrac{2}{x^2} > 0$ for all $x \neq 0$. Therefore, the graph of h is concave up for all $x \neq 0$.

$$3 : \begin{cases} 1 : h''(x) \\ 1 : h''(x) > 0 \\ 1 : \text{answer} \end{cases}$$

(c) $h'(4) = \dfrac{16 - 2}{4} = \dfrac{7}{2}$

$y + 3 = \dfrac{7}{2}(x - 4)$

1 : tangent line equation

(d) The tangent line is below the graph because the graph of h is concave up for $x > 4$.

1 : answer with reason

1997 AB4

Let f be the function given by $f(x) = x^3 - 6x^2 + p$, where p is an arbitrary constant.

(a) Write an expression for $f'(x)$ and use it to find the relative maximum and minimum values of f in terms of p. Show the analysis that leads to your conclusion.

(b) For what values of the constant p does f have 3 distinct roots?

(c) Find the value of p such that the average value of f over the closed interval $[-1, 2]$ is 1.

1993 AB4/BC3

Let f be the function defined by $f(x) = \ln(2 + \sin x)$ for $\pi \le x \le 2\pi$.

(a) Find the absolute maximum value and the absolute minimum value of f. Show the analysis that leads to your conclusion.

(b) Find the x-coordinate of each inflection point on the graph of f. Justify your answer.

1993 AB1

Let f be the function given by $f(x) = x^3 - 5x^2 + 3x + k$, where k is a constant.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave downward?

(c) Find the value of k for which f has 11 as its relative minimum.

1992 BC4

Let f be a function defined by $f(x) = \begin{cases} 2x - x^2 & \text{for } x \le 1, \\ x^2 + kx + p & \text{for } x > 1. \end{cases}$

(a) For what values of k and p will f be continuous and differentiable at $x = 1$?

(b) For the values of k and p found in part (a), on what interval or intervals is f increasing?

(c) Using the values of k and p found in part (a), find all points of inflection of the graph of f. Support your conclusion.

1992 AB1

Let f be the function defined by $f(x) = 3x^5 - 5x^3 + 2$.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave upward?

(c) Write the equation of each horizontal tangent line to the graph of f.

1991 AB5

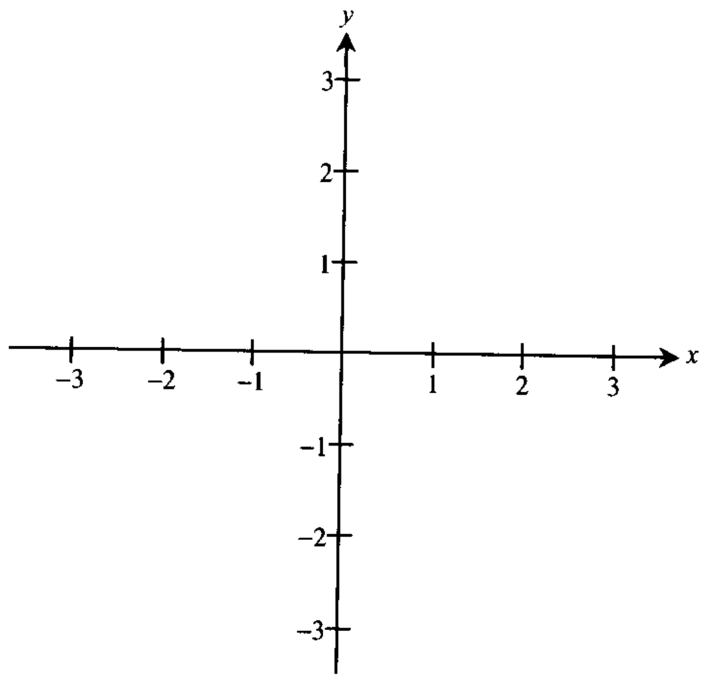
Let f be a function that is <u>even</u> and continuous on the closed interval $[-3,3]$. The function f and its derivatives have the properties indicated in the table below.

x	0	$0<x<1$	1	$1<x<2$	2	$2<x<3$
$f(x)$	1	Positive	0	Negative	-1	Negative
$f'(x)$	Undefined	Negative	0	Negative	Undefined	Positive
$f''(x)$	Undefined	Positive	0	Negative	Undefined	Negative

(a) Find the x-coordinate of each point at which f attains an absolute maximum value or an absolute minimum value. For each x-coordinate you give, state whether f attains an absolute maximum or an absolute minimum.

(b) Find the x-coordinate of each point of inflection on the graph of f. Justify your answer.

(c) In the xy-plane provided below, sketch the graph of a function with all the given characteristics of f.



1990 AB5

Let f be the function defined by $f(x) = \sin^2 x - \sin x$ for $0 \le x \le \frac{3\pi}{2}$.

(a) Find the x-intercepts of the graph of f.

(b) Find the intervals on which f is increasing.

(c) Find the absolute maximum value and the absolute minimum value of f. Justify your answer.

1989 BC3

Consider the function f defined by $f(x) = e^x \cos x$ with domain $[0, 2\pi]$.

(a) Find the absolute maximum and minimum values of $f(x)$.

(b) Find the intervals on which f is increasing.

(c) Find the x-coordinate of each point of inflection of the graph of f.

6. Let f be the function whose graph goes through the point $(3,6)$ and whose derivative is given by $f'(x) = \dfrac{1+e^x}{x^2}$.

(a) Write an equation of the line tangent to the graph of f at $x = 3$ and use it to approximate $f(3.1)$.

(b) Use Euler's method, starting at $x = 3$ with a step size of 0.05, to approximate $f(3.1)$. Use f'' to explain why this approximation is less than $f(3.1)$.

(c) Use $\int_3^{3.1} f'(x)\,dx$ to evaluate $f(3.1)$.

(a) $f'(3) = \dfrac{1+e^3}{9} = 2.342$ or 2.343

$$y - 6 = \frac{1+e^3}{9}(x-3)$$

$$y = 6 + \frac{1+e^3}{9}(x-3)$$

$$f(3.1) \approx 6 + \frac{1+e^3}{9}(0.1) = 6.234$$

$3 \begin{cases} 1\text{: } f'(3) \\ 1\text{: equation} \\ 1\text{: approximation of } f(3.1) \end{cases}$

(b) $f(3.05) \approx f(3) + f'(3)(0.05)$

$= 6 + 0.11714 = 6.11714$

$f(3.1) \approx 6.11714 + f'(3.05)(0.05)$

$= 6.11714 + (2.37735)(0.05) = 6.236$

$$f''(x) = \frac{x^2e^x - 2x(1+e^x)}{x^4} = \frac{(x-2)e^x - 2}{x^3}$$

For $x \geq 3$, $f''(x) > \dfrac{e^x - 2}{x^3} > 0$ and the graph of f is concave upward on $(3, 3.1)$. Therefore, the Euler approximation lines at 3 and 3.05 lie below the graph.

$4 \begin{cases} 1\text{: Euler's method equations or equivalent table} \\ 1\text{: Euler approximation to } f(3.1) \text{ (not eligible without first point)} \\ 1\text{: } f''(x) \\ 1\text{: reason} \end{cases}$

(c) $f(3.1) - f(3) = \displaystyle\int_3^{3.1} \frac{1+e^x}{x^2}\,dx$

$f(3.1) = 6 + 0.2378 = 6.237$ or 6.238

$2 \begin{cases} 1\text{: } \displaystyle\int_3^{3.1} \frac{1+e^x}{x^2}\,dx = f(3.1) - f(3) \\ 1\text{: answer} \end{cases}$

1989 BC4

Consider the curve given by the parametric equations

$$x = 2t^3 - 3t^2 \quad \text{and} \quad y = t^3 - 12t$$

(a) In terms of t, find $\frac{dy}{dx}$.

(b) Write an equation for the line tangent to the curve at the point where $t = -1$.

(c) Find the x- and y-coordinates for each critical point on the curve and identify each point as having a vertical or horizontal tangent.

1994 AB 3

Consider the curve defined by $x^2 + xy + y^2 = 27$.

(a) Write an expression for the slope of the curve at any point (x, y).

(b) Determine whether the lines tangent to the curve at the x-intercepts of the curve are parallel. Show the analysis that leads to your conclusion.

(c) Find the points on the curve where the lines tangent to the curve are vertical.

1994 AB 1

Let f be the function given by $f(x) = 3x^4 + x^3 - 21x^2$.

(a) Write an equation of the line tangent to the graph of f at the point $(2, -28)$.

(b) Find the absolute minimum value of f. Show the analysis that leads to your conclusion.

(c) Find the x-coordinate of each point of inflection on the graph of f. Show the analysis that leads to your conclusion.

1997 BC4

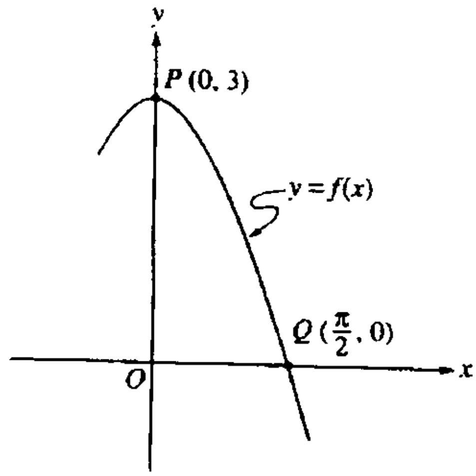
Let $x = ky^2 + 2$, where $k > 0$.

(a) Show that for all $k > 0$, the point $\left(4, \sqrt{\frac{2}{k}}\right)$ is on the graph of $x = ky^2 + 2$.

(b) Show that for all $k > 0$, the tangent line to the graph of $x = ky^2 + 2$ at the point $\left(4, \sqrt{\frac{2}{k}}\right)$ passes through the origin.

(c) Let R be the region in the first quadrant bounded by the x-axis, the graph of $x = ky^2 + 2$, and the line $x = 4$. Write an integral expression for the area of the region R and show that this area decreases as k increases.

1997 AB2



Let f be the function given by $f(x) = 3\cos x$. As shown above, the graph of f crosses the y-axis at point P and the x-axis at point Q.

(a) Write an equation for the line passing through the points P and Q.

(b) Write an equation for the line tangent to the graph of f at point Q. Show the analysis that leads to your conclusion.

(c) Find the x-coordinate of the point on the graph of f, between points P and Q, at which the line tangent to the graph of f is parallel to line PQ.

(d) Let R be the region in the first quadrant bounded by the graph of f and line segment PQ. Write an integral expression for the volume of the solid generated by revolving the region R about the x-axis. Do not evaluate.

1995 AB3

Consider the curve defined by $-8x^2 + 5xy + y^3 = -149$.

(a) Find $\frac{dy}{dx}$.

(b) Write an equation for the line tangent to the curve at the point $(4, -1)$.

(c) There is a number k so that the point $(4.2, k)$ is on the curve. Using the tangent line found in part (b), approximate the value of k.

(d) Write an equation that can be solved to find the actual value of k so that the point $(4.2, k)$ is on the curve.

(e) Solve the equation found in part (d) for the value of k.

1992 AB4/BC1

Consider the curve defined by the equation $y + \cos y = x + 1$ for $0 \leq y \leq 2\pi$.

(a) Find $\dfrac{dy}{dx}$ in terms of y.

(b) Write an equation for each vertical tangent to the curve.

(c) Find $\dfrac{d^2y}{dx^2}$ in terms of y.

1991 AB3

Let f be the function defined by $f(x) = (1 + \tan x)^{\frac{3}{2}}$ for $-\frac{\pi}{4} < x < \frac{\pi}{2}$.

(a) Write an equation for the line tangent to the graph of f at the point where $x = 0$.

(b) Using the equation found in part (a), approximate $f(0.02)$.

(c) Let f^{-1} denote the inverse function of f. Write an expression that gives $f^{-1}(x)$ for all x in the domain of f^{-1}.

1991 AB1

Let f be the function that is defined for all real numbers x and that has the following properties.

(i) $f''(x) = 24x - 18$

(ii) $f'(1) = -6$

(iii) $f(2) = 0$

(a) Find each x such that the line tangent to the graph of f at $(x, f(x))$ is horizontal.

(b) Write an expression for $f(x)$.

(c) Find the average value of f on the interval $1 \leq x \leq 3$.

1989 BC1

Let f be a function such that $f''(x) = 6x + 8$.

(a) Find $f(x)$ if the graph of f is tangent to the line $3x - y = 2$ at the point $(0, -2)$.

(b) Find the average value of $f(x)$ on the closed interval $[-1, 1]$.

1989 AB1

Let f be the function given by $f(x) = x^3 - 7x + 6$.

(a) Find the zeros of f.

(b) Write an equation of the line tangent to the graph of f at $x = -1$.

(c) Find the number c that satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[1, 3]$.

AP Calculus BC–6

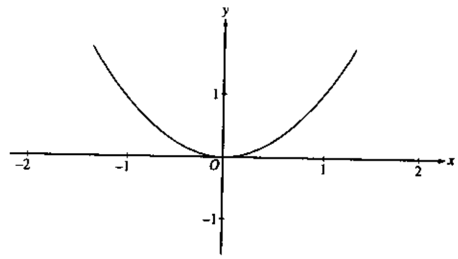
Consider the differential equation given by $\frac{dy}{dx} = x(y-1)^2$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eleven points indicated.

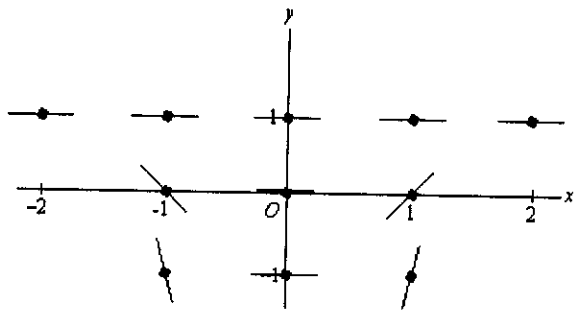
(b) Use the slope field for the given differential equation to explain why a solution could not have the graph shown below.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -1$.

(d) Find the range of the solution found in part (c).



(a)



$2 \begin{cases} 1: \text{ zero slope at 7 points with } \\ \quad y = 1 \text{ and } x = 0 \\ 1: \text{ negative slope at } (-1,0) \text{ and } (-1,-1) \\ \quad \text{positive slope at } (1,0) \text{ and } (1,-1) \\ \quad \text{steeper slope at } y = -1 \text{ than } y = 0 \end{cases}$

(b) The graph does not have slope 0 where $y = 1$.

– or –

The slope field shown suggests that solutions are asymptotic to $y = 1$ from below, but the graph does not exhibit this behavior.

1 : reason

(c) $\frac{1}{(y-1)^2}\,dy = x\,dx$

$-\frac{1}{y-1} = \frac{1}{2}x^2 + C$

$\frac{1}{2} = 0 + C; \quad C = \frac{1}{2}$

$-\frac{1}{y-1} = \frac{1}{2}(x^2+1); \quad y = 1 - \frac{2}{x^2+1}$

$5 \begin{cases} 1: \text{ separates variables} \\ 1: \text{ antiderivatives} \\ 1: \text{ constant of integration} \\ 1: \text{ uses initial condition } f(0) = -1 \\ 1: \text{ solves for } y \\ \quad 0/1 \text{ if } y \text{ is linear} \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(d) range is $-1 \le y < 1$

1 : answer

0/1 if -1 not in range

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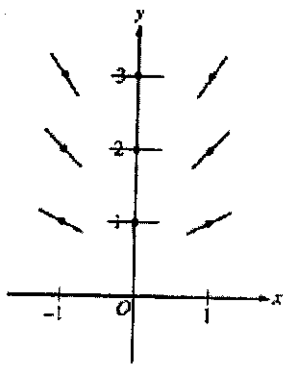
4. Consider the differential equation given by $\dfrac{dy}{dx} = \dfrac{xy}{2}$.
   (a) On the axes provided below, sketch a slope field for the given differential equation at the nine points indicated.
   (b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$, with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.
   (c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

(a)



1: line segments at nine points with negative - zero - positive slope left to right and increasing steepness bottom to top at $x = 1$ and $x = -1$

(b) $f(0.1) \approx f(0) + f'(0)(0.1)$

$= 3 + \frac{1}{2}(0)(3)(0.1) = 3$

$f(0.2) \approx f(0.1) + f'(0.1)(0.1)$

$= 3 + \frac{1}{2}(0.1)(3)(0.1)$

$= 3 + \frac{.03}{2} = 3.015$

2 { 1: Euler's Method equations or equivalent table; 1: answer (not eligible without first point) }

Special Case: 1/2 for first iteration 3.015 and second iteration 3.045

(c) $\dfrac{dy}{dx} = \dfrac{xy}{2}$

$\displaystyle\int \frac{dy}{y} = \int \frac{x}{2}\,dx$

$\ln|y| = \frac{1}{4}x^2 + C_1$

$y = Ce^{x^2/4}$

$3 = Ce^0 \implies C = 3$

$y = 3e^{x^2/4}$

$f(0.2) = 3e^{.04/4} = 3e^{.01} = 3.030$

6 {
1: separates variables
1: antiderivative of dy term
1: antiderivative of dx term
1: solves for y
1: solves for constant of integration
1: evaluates $f(0.2)$
}

Note: max 4/6 [1-1-1-0-0-1] if no constant of integration

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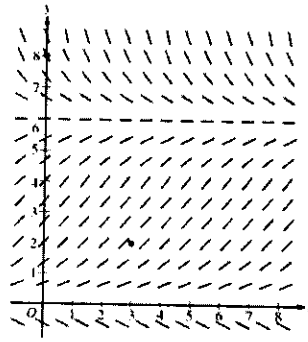
Question 6

Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.
(Note: Use the axes provided in the exam booklet.)

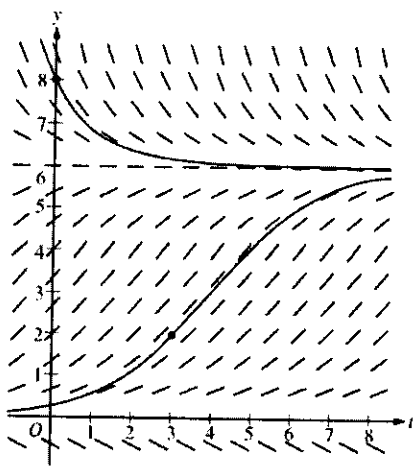


(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

(c) Write the second-degree Taylor polynomial for f about $t = 0$, and use it to approximate $f(1)$.

(d) What is the range of f for $t \geq 0$?

(a)



$2 : \begin{cases} 1\text{: solution curve through } (0,8) \\ 1\text{: solution curve through } (3,2) \end{cases}$

(b) $f\left(\frac{1}{2}\right) \approx 8 + (-2)\left(\frac{1}{2}\right) = 7$

$f(1) \approx 7 + \left(-\frac{7}{8}\right)\left(\frac{1}{2}\right) = \frac{105}{16}$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{approximation of } f(1) \end{cases}$

(c) $\frac{d^2y}{dt^2} = \frac{1}{8}\frac{dy}{dt}(6 - y) + \frac{y}{8}\left(-\frac{dy}{dt}\right)$

$f(0) = 8$; $f'(0) = \left.\frac{dy}{dt}\right|_{t=0} = \frac{8}{8}(6 - 8) = -2$; and

$f''(0) = \left.\frac{d^2y}{dt^2}\right|_{t=0} = \frac{1}{8}(-2)(-2) + \frac{8}{8}(2) = \frac{5}{2}$

The second-degree Taylor polynomial for f about $t = 0$ is $P_2(t) = 8 - 2t + \frac{5}{4}t^2$.

$f(1) \approx P_2(1) = \frac{29}{4}$

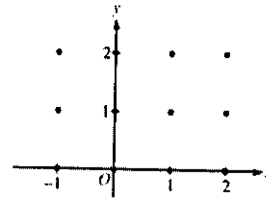
$4 : \begin{cases} 2 : \frac{d^2y}{dt^2} \\ 1 : \text{second-degree Taylor polynomial} \\ 1 : \text{approximation of } f(1) \end{cases}$

(d) The range of f for $t \geq 0$ is $6 < y \leq 8$.

1 : answer

Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.



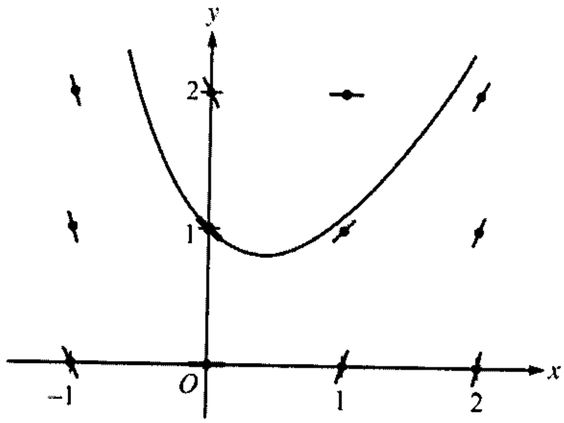
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(0, 1)$. **(Note: Use the axes provided in the pink test booklet.)**

(b) The solution curve that passes through the point $(0, 1)$ has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the y-coordinate of this local minimum?

(c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

(d) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.

(a)



$3: \begin{cases} 1: \text{zero slopes} \\ 1: \text{nonzero slopes} \\ 1: \text{curve through } (0, 1) \end{cases}$

(b) $\frac{dy}{dx} = 0$ when $2x = y$

The y-coordinate is $2\ln\left(\frac{3}{2}\right)$.

$2: \begin{cases} 1: \text{sets } \frac{dy}{dx} = 0 \\ 1: \text{answer} \end{cases}$

(c) $f(-0.2) \approx f(0) + f'(0)(-0.2)$
$= 1 + (-1)(-0.2) = 1.2$
$f(-0.4) \approx f(-0.2) + f'(-0.2)(-0.2)$
$\approx 1.2 + (-1.6)(-0.2) = 1.52$

$2: \begin{cases} 1: \text{Euler's method with two steps} \\ 1: \text{Euler approximation to } f(-0.4) \end{cases}$

(d) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - 2x + y$

$\frac{d^2y}{dx^2}$ is positive in quadrant II because $x < 0$ and $y > 0$.

$1.52 < f(-0.4)$ since all solution curves in quadrant II are concave up.

$2: \begin{cases} 1: \frac{d^2y}{dx^2} \\ 1: \text{answer with reason} \end{cases}$

1994 AB 4

A particle moves along the x-axis so that at any time $t > 0$ its velocity is given by $v(t) = t\ln t - t$. At time $t = 1$, the position of the particle is $x(1) = 6$.

(a) Write an expression for the acceleration of the particle.

(b) For what values of t is the particle moving to the right?

(c) What is the minimum velocity of the particle? Show the analysis that leads to your conclusion.

(d) Write an expression of the position $x(t)$ of the particle.

1994 BC 3

A particle moves along the graph of $y = \cos x$ so that the x-coordinate of acceleration is always 2. At time $t = 0$, the particle is at the point $(\pi, -1)$ and the velocity vector of the particle is $(0, 0)$.

(a) Find the x- and y-coordinates of the position of the particle in terms of t.

(b) Find the speed of the particle when its position is $(4, \cos 4)$.

1990 AB1

A particle, initially at rest, moves along the x-axis so that its acceleration at any time $t \geq 0$ is given by $a(t) = 12t^2 - 4$. The position of the particle when $t = 1$ is $x(1) = 3$.

(a) Find the values of t for which the particle is at rest.

(b) Write an expression for the position $x(t)$ of the particle at any time $t \geq 0$.

(c) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

1989 AB3

A particle moves along the x-axis in such a way that its acceleration at time t for $t \geq 0$ is given by $a(t) = 4\cos(2t)$. At time $t = 0$, the velocity of the particle is $v(0) = 1$ and its position is $x(0) = 0$.

(a) Write an equation for the velocity $v(t)$ of the particle.

(b) Write an equation for the position $x(t)$ of the particle.

(c) For what values of t, $0 \leq t \leq \pi$, is the particle at rest?

1997 AB6/BC6

Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt} = -2v - 32$, with initial condition $v(0) = -50$.

(a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

(b) Terminal velocity is defined as $\lim_{t \to \infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

(c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

1997 AB1

A particle moves along the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 3t^2 - 2t - 1$. The position $x(t)$ is 5 for $t = 2$.

(a) Write a polynomial expression for the position of the particle at any time $t \geq 0$.

(b) For what values of t, $0 \leq t \leq 3$, is the particle's instantaneous velocity the same as its average velocity on the closed interval $[0,3]$?

(c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.

1995 AB2

A particle moves along the y-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = t \cos t$. At time $t = 0$, the position of the particle is $y = 3$.

(a) For what values of t, $0 \leq t \leq 5$, is the particle moving upward?

(b) Write an expression for the acceleration of the particle in terms of t.

(c) Write an expression for the position $y(t)$ of the particle.

(d) For $t > 0$, find the position of the particle the first time the velocity of the particle is zero.

1993 BC2

The position of a particle at any time $t \geq 0$ is given by $x(t) = t^2 - 3$ and $y(t) = \frac{2}{3}t^3$.

(a) Find the magnitude of the velocity vector at $t = 5$.

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 5$.

(c) Find $\frac{dy}{dx}$ as a function of x.

1993 AB2

A particle moves on the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = 2te^{-t}$.

(a) Find the acceleration of the particle at $t = 0$.

(b) Find the velocity of the particle when its acceleration is 0.

(c) Find the total distance traveled by the particle from $t = 0$ to $t = 5$.

1992 BC3

At time t, $0 \le t \le 2\pi$, the position of a particle moving along a path in the xy-plane is given by the parametric equations $x = e^t \sin t$ and $y = e^t \cos t$.

(a) Find the slope of the path of the particle at time $t = \frac{\pi}{2}$.

(b) Find the speed of the particle when $t = 1$.

(c) Find the distance traveled by the particle along the path from $t = 0$ to $t = 1$.

1992 AB2

A particle moves along the x-axis so that its velocity at time t, $0 \le t \le 5$, is given by $v(t) = 3(t-1)(t-3)$. At time $t = 2$, the position of the particle is $x(2) = 0$.

(a) Find the minimum acceleration of the particle.

(b) Find the total distance traveled by the particle.

(c) Find the average velocity of the particle over the interval $0 \le t \le 5$.

1991 BC1

A particle moves on the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 12t^2 - 36t + 15$. At $t = 1$, the particle is at the origin.

(a) Find the position $x(t)$ of the particle at any time $t \geq 0$.

(b) Find all values of t for which the particle is at rest.

(c) Find the maximum velocity of the particle for $0 \leq t \leq 2$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

1990 BC1

A particle starts at time $t = 0$ and moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = (t-1)^3(2t-3)$.

(a) Find the velocity of the particle at any time $t \geq 0$.

(b) For what values of t is the velocity of the particle less than zero?

(c) Find the value of t when the particle is moving and the acceleration is zero.

Question 1

A particle moves in the xy-plane so that its position at any time t, for $-\pi \le t \le \pi$, is given by $x(t) = \sin(3t)$ and $y(t) = 2t$.

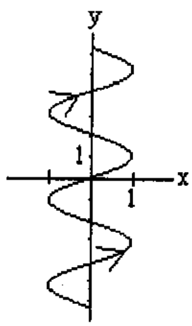
(a) Sketch the path of the particle in the xy-plane provided. Indicate the direction of motion along the path.

(b) Find the range of $x(t)$ and the range of $y(t)$.

(c) Find the smallest positive value of t for which the x-coordinate of the particle is a local maximum. What is the speed of the particle at this time?

(d) Is the distance traveled by the particle from $t = -\pi$ to $t = \pi$ greater than 5π? Justify your answer.

(a)



$2 \begin{cases} 1 : \text{graph} \\ \quad \text{three cycles of sine} \\ \quad x \text{ between } -1 \text{ and } 1 \\ \quad y \text{ between } -2\pi \text{ and } 2\pi \\ 1 : \text{direction} \end{cases}$

(b) $-1 \le x(t) \le 1$

$-2\pi \le y(t) \le 2\pi$

$2 \begin{cases} 1 : \text{closed interval for } x(t) \\ 1 : \text{closed interval for } y(t) \end{cases}$

(c) $x'(t) = 3\cos 3t = 0$

$3t = \frac{\pi}{2};\ t = \frac{\pi}{6}$

Speed $= \sqrt{9\cos^2(3t) + 4}$

At $t = \frac{\pi}{6}$,

Speed $= \sqrt{9\cos^2\left(\frac{\pi}{2}\right) + 4} = 2$

$3 \begin{cases} 1 : x'(t) = 3\cos 3t = 0 \\ 1 : \text{solves for } t \\ 1 : \text{speed at student's time} \end{cases}$

(d) Distance $= \int_{-\pi}^{\pi} \sqrt{9\cos^2(3t) + 4}\, dt$

$= 17.973 > 5\pi$

$2 \begin{cases} 1 : \text{integral for distance} \\ 1 : \text{conclusion with justification} \end{cases}$

A moving particle has position $(x(t), y(t))$ at time t. The position of the particle at time $t = 1$ is (2,6) and the velocity vector at any time $t > 0$ is given by $\left(1 - \frac{1}{t^2}, 2 + \frac{1}{t^2}\right)$.

(a) Find the acceleration vector at time $t = 3$.

(b) Find the position of the particle at time $t = 3$.

(c) For what time $t > 0$ does the line tangent to the path of the particle at $(x(t), y(t))$ have a slope of 8?

(d) The particle approaches a line as $t \to \infty$. Find the slope of this line. Show the work that leads to your conclusion.

(a) acceleration vector $= (x''(t), y''(t)) = \left(\frac{2}{t^3}, -\frac{2}{t^3}\right)$

$(x''(3), y''(3)) = \left(\frac{2}{27}, -\frac{2}{27}\right)$

$2\begin{cases} 1: \text{ components of acceleration vector as a function of } t \\ 1: \text{ acceleration vector at } t = 3 \end{cases}$

(b) $(x(t), y(t)) = \left(t + \frac{1}{t} + C_1,\ 2t - \frac{1}{t} + C_2\right)$

$(2,6) = (x(1), y(1)) = (2 + C_1, 1 + C_2)$

$C_1 = 0,\ C_2 = 5$

$(x(3), y(3)) = \left(3 + \frac{1}{3}, 6 - \frac{1}{3} + 5\right) = \left(\frac{10}{3}, \frac{32}{3}\right)$

$3\begin{cases} 1: \text{ antidifferentiation} \\ 1: \text{ uses initial condition at } t = 1 \\ 1: \text{ position at } t = 3 \end{cases}$

Note: max 1/3 [1–0–0] if no constants of integration

(c) $\dfrac{dy}{dx} = \dfrac{2 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = 8$

$2 + \frac{1}{t^2} = 8\left(1 - \frac{1}{t^2}\right); \quad t^2 = \frac{9}{6}$

$t = \sqrt{\frac{3}{2}}$

$2\begin{cases} 1: \ \frac{dy}{dx} = 8 \text{ as equation in } t \\ 1: \text{ solution for } t \end{cases}$

(d) $\displaystyle\lim_{t\to\infty} \frac{dy}{dx} = \lim_{t\to\infty} \frac{2 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = 2$

– or –

Since $x(t) \to \infty$ as $t \to \infty$, the slope of the line is

$\displaystyle\lim_{t\to\infty} \frac{y(t)}{x(t)} = \lim_{t\to\infty} \frac{2t - \frac{1}{t} + 5}{t + \frac{1}{t}} = 2$

$2\begin{cases} 1: \text{ considers limit of } \frac{dy}{dx} \text{ or } \frac{y(t)}{x(t)} \\ 1: \text{ answer} \end{cases}$

Note: 0/2 if no consideration of limit

1998 Calculus BC Scoring Guidelines

6. A particle moves along the curve defined by the equation $y = x^3 - 3x$. The x–coordinate of the particle, $x(t)$, satisfies the equation $\frac{dx}{dt} = \frac{1}{\sqrt{2t+1}}$, for $t \geq 0$ with initial condition $x(0) = -4$.
   (a) Find $x(t)$ in terms of t.
   (b) Find $\frac{dy}{dt}$ in terms of t.
   (c) Find the location and speed of the particle at time $t = 4$.

(a) $x(t) = \int \frac{1}{\sqrt{2t+1}}\,dt$

$x(t) = \sqrt{2t+1} + C$

$x(0) = -4 = 1 + C \implies C = -5$

$x(t) = \sqrt{2t+1} - 5$

$$3 \begin{cases} 1\text{: } x(t) = \int \frac{dt}{\sqrt{2t+1}} \\ 1\text{: } x(t) = \sqrt{2t+1} + C \\ 1\text{: evaluates } C \end{cases}$$

(b) $y = x^3 - 3x$

$$\frac{dy}{dt} = 3x^2\frac{dx}{dt} - 3\frac{dx}{dt}$$

$$= (3x^2 - 3)\frac{dx}{dt}$$

$$= \left[3\left(\sqrt{2t+1} - 5\right)^2 - 3\right]\left[\frac{1}{\sqrt{2t+1}}\right]$$

2: answer

$<-1>$ each error

Note: failure to express $\frac{dy}{dt}$ solely in terms of t is a single error

(c) $x(4) = \sqrt{9} - 5 = -2$

$y(4) = (-2)^3 - 3(-2) = -2$

Location at $t = 4$ is $(-2, -2)$

$$\left.\frac{dx}{dt}\right|_{t=4} = \frac{1}{3}$$

$$\left.\frac{dy}{dt}\right|_{t=4} = \frac{3(3-5)^2 - 3}{3} = 3$$

$$\text{Speed} = \sqrt{\left(\frac{1}{3}\right)^2 + 3^2} = \sqrt{\frac{82}{9}} = 3.018$$

$$4 \begin{cases} 1\text{: position} \\ 1\text{: evaluates } \frac{dx}{dt} \text{ and } \frac{dy}{dt} \text{ at } t = 4 \\ 1\text{: uses speed formula} \\ 1\text{: answer} \end{cases}$$

Question 3

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sin^{-1}\left(1 - 2e^{-t}\right) \text{ and } \frac{dy}{dt} = \frac{4t}{1+t^3}$$

for $t \geq 0$. At time $t = 2$, the object is at the point $(6, -3)$. (Note: $\sin^{-1} x = \arcsin x$)

(a) Find the acceleration vector and the speed of the object at time $t = 2$.

(b) The curve has a vertical tangent line at one point. At what time t is the object at this point?

(c) Let $m(t)$ denote the slope of the line tangent to the curve at the point $(x(t), y(t))$. Write an expression for $m(t)$ in terms of t and use it to evaluate $\lim_{t\to\infty} m(t)$.

(d) The graph of the curve has a horizontal asymptote $y = c$. Write, but do not evaluate, an expression involving an improper integral that represents this value c.

(a) $a(2) = \langle 0.395 \text{ or } 0.396, -0.741 \text{ or } -0.740\rangle$ Speed $= \sqrt{x'(2)^2 + y'(2)^2} = 1.207 \text{ or } 1.208$	$2: \begin{cases} 1: \text{acceleration} \\ 1: \text{speed} \end{cases}$
(b) $\sin^{-1}\left(1 - 2e^{-t}\right) = 0$ $1 - 2e^{-t} = 0$ $t = \ln 2 = 0.693$ and $\frac{dy}{dt} \neq 0$ when $t = \ln 2$	$2: \begin{cases} 1: x'(t) = 0 \\ 1: \text{answer} \end{cases}$
(c) $m(t) = \frac{4t}{1+t^3} \cdot \frac{1}{\sin^{-1}\left(1 - 2e^{-t}\right)}$ $\lim_{t\to\infty} m(t) = \lim_{t\to\infty}\left(\frac{4t}{1+t^3} \cdot \frac{1}{\sin^{-1}\left(1 - 2e^{-t}\right)}\right)$ $= 0\left(\frac{1}{\sin^{-1}(1)}\right) = 0$	$2: \begin{cases} 1: m(t) \\ 1: \text{limit value} \end{cases}$
(d) Since $\lim_{t\to\infty} x(t) = \infty$, $c = \lim_{t\to\infty} y(t) = -3 + \int_2^{\infty} \frac{4t}{1+t^3}\,dt$	$3: \begin{cases} 1: \text{integrand} \\ 1: \text{limits} \\ 1: \text{initial value consistent} \\ \quad \text{with lower limit} \end{cases}$

Question 3

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \ge 0$ with $\frac{dx}{dt} = 3 + \cos(t^2)$. The derivative $\frac{dy}{dt}$ is not explicitly given. At time $t = 2$, the object is at position $(1, 8)$.

(a) Find the x-coordinate of the position of the object at time $t = 4$.

(b) At time $t = 2$, the value of $\frac{dy}{dt}$ is -7. Write an equation for the line tangent to the curve at the point $(x(2), y(2))$.

(c) Find the speed of the object at time $t = 2$.

(d) For $t \ge 3$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $2t + 1$. Find the acceleration vector of the object at time $t = 4$.

(a) $x(4) = x(2) + \int_2^4 \left(3 + \cos(t^2)\right) dt$

$= 1 + \int_2^4 \left(3 + \cos(t^2)\right) dt = 7.132 \text{ or } 7.133$

$3 : \begin{cases} 1 : \int_2^4 \left(3 + \cos(t^2)\right) dt \\ 1 : \text{handles initial condition} \\ 1 : \text{answer} \end{cases}$

(b) $\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=2} = \frac{-7}{3 + \cos 4} = -2.983$

$y - 8 = -2.983(x - 1)$

$2 : \begin{cases} 1 : \text{finds } \left.\frac{dy}{dx}\right|_{t=2} \\ 1 : \text{equation} \end{cases}$

(c) The speed of the object at time $t = 2$ is $\sqrt{(x'(2))^2 + (y'(2))^2} = 7.382 \text{ or } 7.383.$

1 : answer

(d) $x''(4) = 2.303$

$y'(t) = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (2t + 1)\left(3 + \cos(t^2)\right)$

$y''(4) = 24.813 \text{ or } 24.814$

The acceleration vector at $t = 4$ is $\langle 2.303, 24.813 \rangle$ or $\langle 2.303, 24.814 \rangle$.

$3 : \begin{cases} 1 : x''(4) \\ 1 : \frac{dy}{dt} \\ 1 : \text{answer} \end{cases}$

Question 1

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with

$$\frac{dx}{dt} = \cos\left(t^3\right) \text{ and } \frac{dy}{dt} = 3\sin\left(t^2\right)$$

for $0 \le t \le 3$. At time $t = 2$, the object is at position (4,5).

(a) Write an equation for the line tangent to the curve at (4,5).

(b) Find the speed of the object at time $t = 2$.

(c) Find the total distance traveled by the object over the time interval $0 \le t \le 1$.

(d) Find the position of the object at time $t = 3$.

(a) $\dfrac{dy}{dx} = \dfrac{3\sin\left(t^2\right)}{\cos\left(t^3\right)}$

$\left.\dfrac{dy}{dx}\right|_{t=2} = \dfrac{3\sin\left(2^2\right)}{\cos\left(2^3\right)} = 15.604$

$y - 5 = 15.604(x - 4)$

1 : tangent line

(b) Speed $= \sqrt{\cos^2(8) + 9\sin^2(4)} = 2.275$

1 : answer

(c) Distance $= \displaystyle\int_0^1 \sqrt{\cos^2\left(t^3\right) + 9\sin^2\left(t^2\right)}\, dt$

$= 1.458$

$$3 : \begin{cases} 2 : \text{distance integral} \\ \quad < -1 > \text{ each integrand error} \\ \quad < -1 > \text{ error in limits} \\ 1 : \text{answer} \end{cases}$$

(d) $x(3) = 4 + \displaystyle\int_2^3 \cos\left(t^3\right) dt = 3.953 \text{ or } 3.954$

$y(3) = 5 + \displaystyle\int_2^3 3\sin\left(t^2\right) dt = 4.906$

$$4 : \begin{cases} 1 : \text{definite integral for } x \\ 1 : \text{answer for } x(3) \\ 1 : \text{definite integral for } y \\ 1 : \text{answer for } y(3) \end{cases}$$

Question 4

A particle moves in the xy-plane so that the position of the particle at any time t is given by

$$x(t) = 2e^{3t} + e^{-7t} \text{ and } y(t) = 3e^{3t} - e^{-2t}.$$

(a) Find the velocity vector for the particle in terms of t, and find the speed of the particle at time $t = 0$.

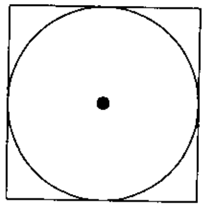
(b) Find $\frac{dy}{dx}$ in terms of t, and find $\lim_{t\to\infty} \frac{dy}{dx}$.

(c) Find each value t at which the line tangent to the path of the particle is horizontal, or explain why none exists.

(d) Find each value t at which the line tangent to the path of the particle is vertical, or explain why none exists.

(a) $x'(t) = 6e^{3t} - 7e^{-7t}$ $y'(t) = 9e^{3t} + 2e^{-2t}$ Velocity vector is $< 6e^{3t} - 7e^{-7t}, 9e^{3t} + 2e^{-2t} >$ Speed $= \sqrt{x'(0)^2 + y'(0)^2} = \sqrt{(-1)^2 + 11^2}$ $= \sqrt{122}$	$3 : \begin{cases} 1 : x'(t) \\ 1 : y'(t) \\ 1 : \text{speed} \end{cases}$
(b) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}}$ $\lim_{t\to\infty} \frac{dy}{dx} = \lim_{t\to\infty} \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}} = \frac{9}{6} = \frac{3}{2}$	$2 : \begin{cases} 1 : \frac{dy}{dx} \text{ in terms of } t \\ 1 : \text{limit} \end{cases}$
(c) Need $y'(t) = 0$, but $9e^{3t} + 2e^{-2t} > 0$ for all t, so none exists.	$2 : \begin{cases} 1 : \text{considers } y'(t) = 0 \\ 1 : \text{explains why none exists} \end{cases}$
(d) Need $x'(t) = 0$ and $y'(t) \neq 0$. $6e^{3t} = 7e^{-7t}$ $e^{10t} = \frac{7}{6}$ $t = \frac{1}{10}\ln\left(\frac{7}{6}\right)$	$2 : \begin{cases} 1 : \text{considers } x'(t) = 0 \\ 1 : \text{solution} \end{cases}$

1994 AB 5-BC 2

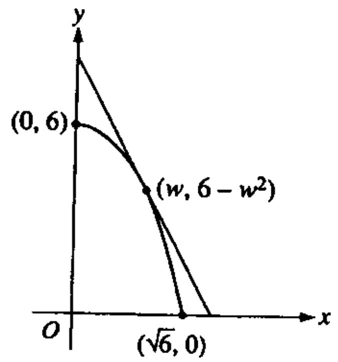


A circle is inscribed in a square as shown in the figure above. The circumference of the circle is increasing at a constant rate of 6 inches per second. As the circle expands, the square expands to maintain the condition of tangency. (Note: A circle with radius r has circumference $C = 2\pi r$ and area $A = \pi r^2$)

(a) Find the rate at which the perimeter of the square is increasing. Indicate units of measure.

(b) At the instant when the area of the circle is 25π square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.

1994 BC 4

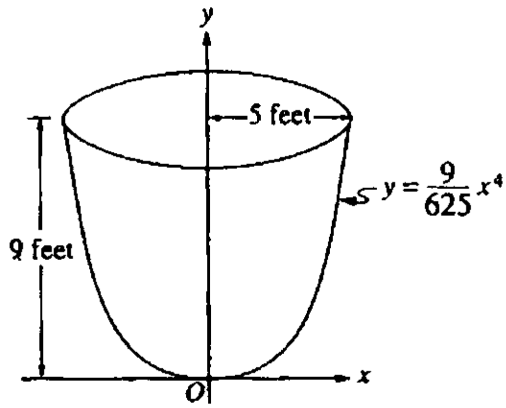


Let $f(x) = 6 - x^2$. For $0 < w < \sqrt{6}$, let $A(w)$ be the area of the triangle formed by the coordinate axes and the line tangent to the graph of f at the point $(w, 6 - w^2)$.

(a) Find $A(1)$.

(b) For what value of w is $A(w)$ a minimum?

1996 BC5

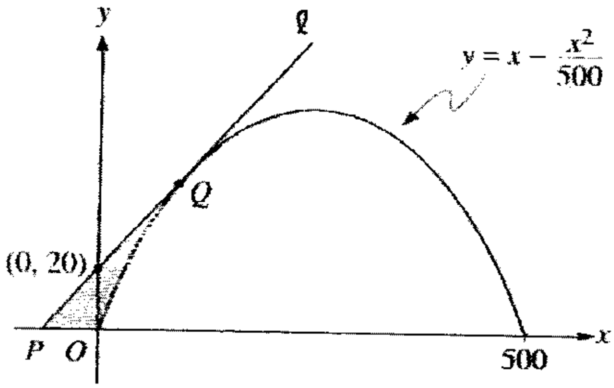


An oil storage tank has the shape as shown above, obtained by revolving the curve $y = \frac{9}{625}x^4$ from $x = 0$ to $x = 5$ about the y-axis, where x and y are measured in feet. Oil weighing 50 pounds per cubic foot flowed into an initially empty tank at a constant rate of 8 cubic feet per minute. When the depth of oil reached 6 feet, the flow stopped.

(a) Let h be the depth, in feet, of oil in the tank. How fast was the depth of oil in the tank increasing when $h = 4$? Indicate units of measure.

(b) Find, to the nearest foot-pound, the amount of work required to empty the tank by pumping all of the oil back to the top of the tank.

1996 AB6



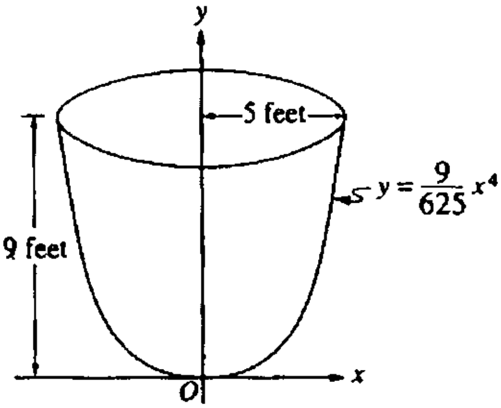
Line ℓ is tangent to the graph of $y = x - \frac{x^2}{500}$ at the point Q, as shown in the figure above.

(a) Find the x-coordinate of point Q.

(b) Write an equation for line ℓ.

(c) Suppose the graph of $y = x - \frac{x^2}{500}$ shown in the figure, where x and y are measured in feet, represents a hill. There is a 50-foot tree growing vertically at the top of the hill. Does a spotlight at point P directed along line ℓ shine on any part of the tree? Show the work that leads to your conclusion.

1996 AB5



An oil storage tank has the shape as shown above, obtained by revolving the curve $y = \frac{9}{625}x^4$ from $x = 0$ to $x = 5$ about the y-axis, where x and y are measured in feet.

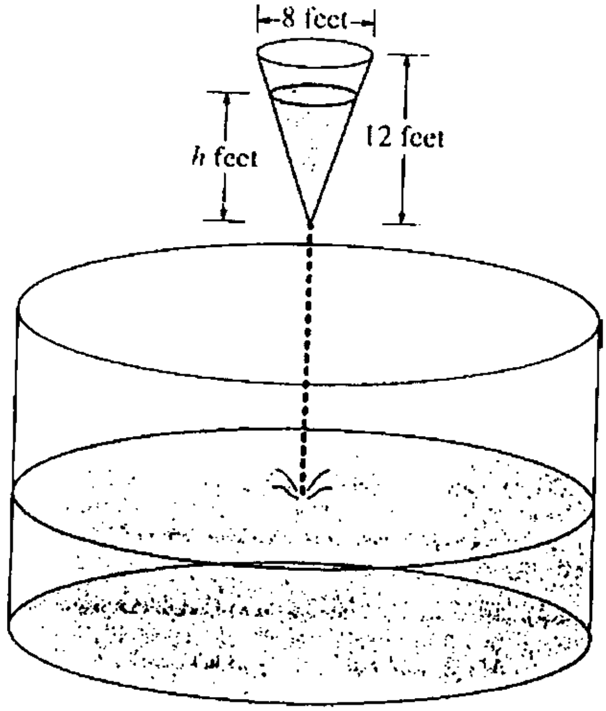
Oil flows into the tank at the constant rate of 8 cubic feet per minute.

(a) Find the volume of the tank. Indicate units of measure.

(b) To the nearest minute, how long would it take to fill the tank if the tank was empty initially?

(c) Let h be the depth, in feet, of oil in the tank. How fast is the depth of oil in the tank increasing when $h = 4$? Indicate units of measure.

1995 AB5/BC3



As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h.

(b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.

(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure.

1992 BC5

The length of a solid cylindrical cord of elastic material is 32 inches. A circular cross section of the cord has radius $\frac{1}{2}$ inch.

(a) What is the volume, in cubic inches, of the cord?

(b) The cord is stretched lengthwise at a constant rate of 18 inches per minute. Assuming that the cord maintains a cylindrical shape and a constant volume, at what rate is the radius of the cord changing one minute after the stretching begins? Indicate units of measure.

(c) A force of $2x$ pounds is required to stretch the cord x inches beyond its natural length of 32 inches. How much work is done during the first minute of stretching described in part (b)? Indicate units of measure.

1990 BC3

Let $f(x) = 12 - x^2$ for $x \geq 0$ and $f(x) \geq 0$.

(a) The line tangent to the graph of f at the point $(k, f(k))$ intercepts the x-axis at $x = 4$. What is the value of k ?

(b) An isosceles triangle whose base is the interval from $(0,0)$ to $(c,0)$ has its vertex on the graph of f. For what value of c does the triangle have maximum area? Justify your answer.

1992 AB6

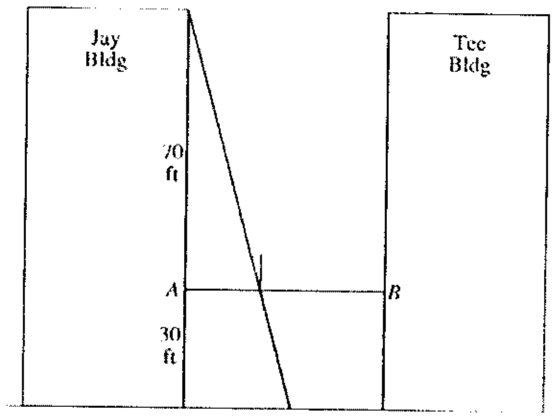
At time t, $t \geq 0$, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At $t = 0$, the radius of the sphere is 1 and at $t = 15$, the radius is 2. (The volume V of a sphere with a radius r is $V = \frac{4}{3}\pi r^3$.)

(a) Find the radius of the sphere as a function of t.

(b) At what time t will the volume of the sphere be 27 times its volume at $t = 0$?

1991 AB6

A tight rope is stretched 30 feet above the ground between the Jay and the Tee buildings, which are 50 feet apart. A tightrope walker, walking at a constant rate of 2 feet per second from point A to point B, is illuminated by a spotlight 70 feet above point A, as shown in the diagram.



(a) How fast is the shadow of the tightrope walker's feet moving along the ground when she is midway between the buildings? (Indicate units of measure.)

(b) How far from point A is the tightrope walker when the shadow of her feet reaches the base of the Tee Building? (Indicate units of measure.)

(c) How fast is the shadow of the tightrope walker's feet moving up the wall of the Tee Building when she is 10 feet from point B? (Indicate units of measure.)

1990 AB4

The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second. (Note: The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.)

(a) At the time when the radius of the sphere is 10 centimeters, what is the rate of increase of its volume?

(b) At the time when the volume of the sphere is 36π cubic centimeters, what is the rate of increase of the area of a cross section through the center of the sphere?

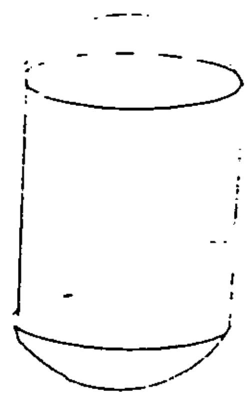
(c) At the time when the volume and the radius of the sphere are increasing at the same numerical rate, what is the radius?

RELATED RATES

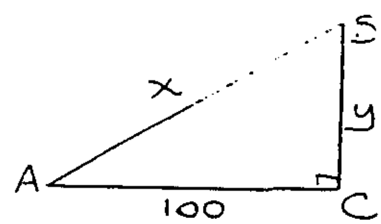
1. A paper cup, which is in the shape of a right circular cone, is 16 cm deep and has a radius of 4 cm. Water is poured into the cup at the constant rate of 2 cm^3/sec.
   (a) At the instant the depth is 5 cm, what is the rate of change of the height?
   (b) At the instant the radius is 3 cm, what is the rate of change of the radius?

2. A snowball is in the shape of a sphere. Its volume is increasing at a constant rate of 10 in^3/min.
   (a) How fast is the radius increasing when the volume is 36π in^3?
   (b) How fast is the surface area increasing when the volume is 36π in^3?

3. (1985) The balloon shown is in the shape of a cylinder with hemispherical ends of the same radius as that of the cylinder. The balloon is being inflated at the rate of 261π cm^3/min. At the instant the radius of the cylinder is 3 cm, the volume of the balloon is 144π cm^3, and the radius of the cylinder is increasing at the rate of 2 cm/min.
   (a) At this instant, what is the height of the cylinder?
   (b) At this instant, how fast is the height of the cylinder increasing?

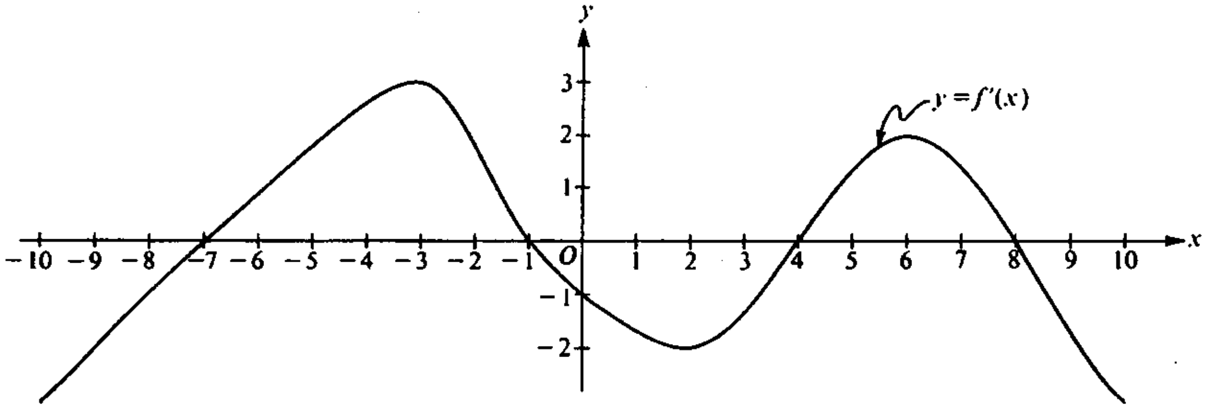


4. (1988) The figure shown represents an observer at point A watching balloon B as it rises from point C. The balloon is rising at a constant rate of 3 m/sec, and the observer is 100 m from point C.
   (a) Find the rate of change in x at the instant when y = 50.
   (b) Find the rate of change in the area of right triangle BCA at the instant when y = 50.
   (c) Find the rate of change in θ at the instant when y = 50.



TURN—>>>

1989 AB5



Note: This is the graph of the derivative of f, not the graph of f.

The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-10 \leq x \leq 10$.

(a) For what values of x does the graph of f have a horizontal tangent?

(b) For what values of x in the interval $(-10,10)$ does f have a relative maximum? Justify your answer.

(c) For value of x is the graph of f concave downward?

1992 AB3

Let f be the function given by $f(x) = \ln\left|\dfrac{x}{1+x^2}\right|$.

(a) Find the domain of f.

(b) Determine whether f is an even function, an odd function, or neither. Justify your conclusion.

(c) At what values of x does f have a relative maximum or a relative minimum? For each such x, use the first derivative test to determine whether $f(x)$ is a relative maximum or a relative minimum.

(d) Find the range of f.

1994 BC 6

Let f and g be functions that are differentiable for all real numbers x and that have the following properties.

(i) $f'(x) = f(x) - g(x)$

(ii) $g'(x) = g(x) - f(x)$

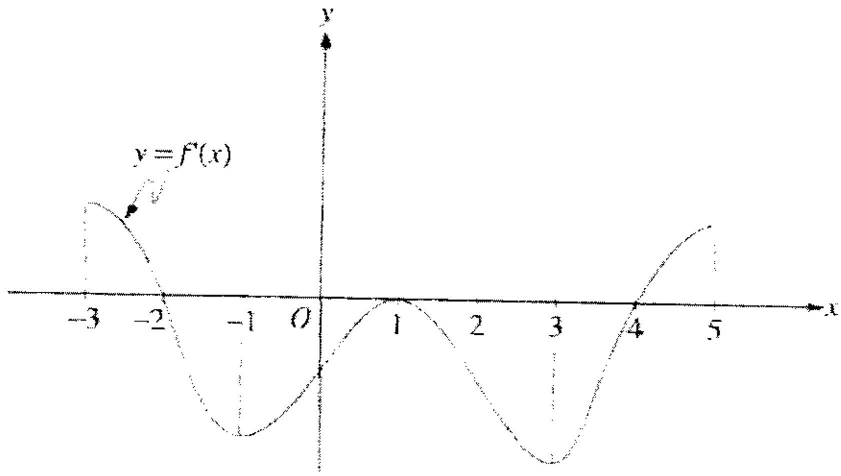
(iii) $f(0) = 5$

(iv) $g(0) = 1$

(a) Prove that $f(x) + g(x) = 6$ for all x.

(b) Find $f(x)$ and $g(x)$. Show your work.

1996 AB1



Note: This is the graph of the derivative of f, not the graph of f.

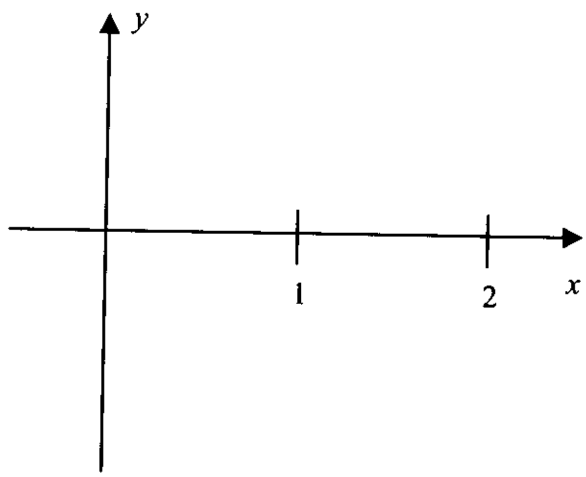
The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-3 < x < 5$.

(a) For what values of x does f have a relative maximum? Why?

(b) For what values of x does f have a relative minimum? Why?

(c) On what intervals is the graph of f concave upward? Use f' to justify your answer.

(d) Suppose that $f(1) = 0$. In the xy-plane provided, draw a sketch that shows the general shape of the graph of the function f on the open interval $0 < x < 2$.



1995 BC5

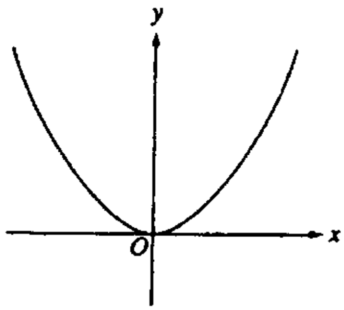


Figure 1
$y = f(x)$

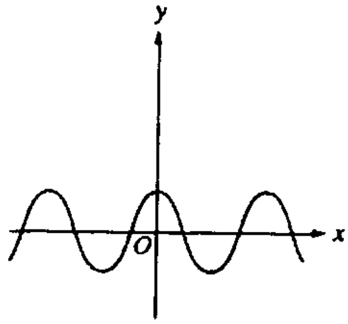


Figure 2
$y = g(x)$

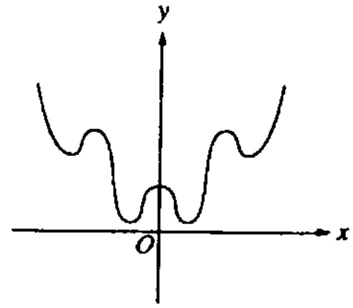
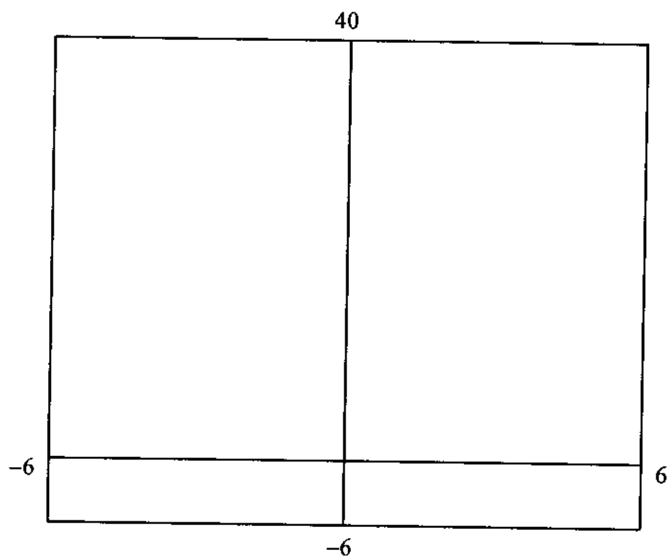


Figure 3

Let $f(x) = x^2$, $g(x) = \cos x$, and $h(x) = x^2 + \cos x$. From the graphs of f and g shown above in Figure 1 and Figure 2, one might think the graph of h should look like the graph in Figure 3.

(a) Sketch the actual graph of h in the viewing window provided below.



Viewing Window
$[-6, 6] \times [-6, 40]$

(b) Use $h''(x)$ to explain why the graph of h does not look like the graph in Figure 3.

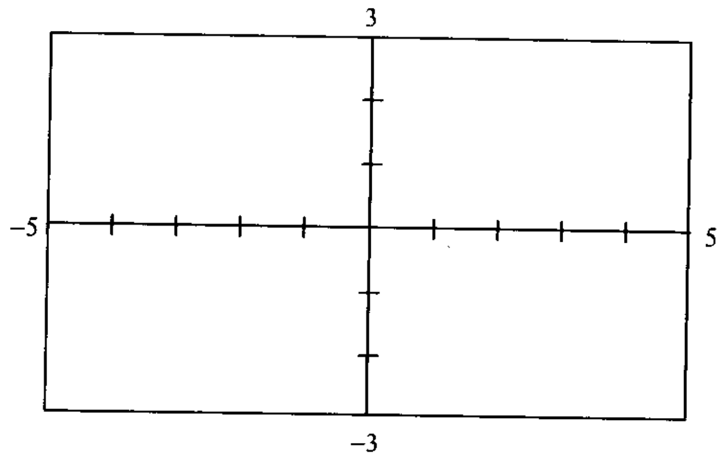
(c) Prove that the graph of $y = x^2 + \cos(kx)$ has either no points of inflection or infinitely many points of inflection, depending on the value of the constant k.

1995 AB1

Let f be the function given by $f(x) = \dfrac{2x}{\sqrt{x^2 + x + 1}}$.

(a) Find the domain of f. Justify your answer.

(b) In the viewing window provided below, sketch the graph of f.



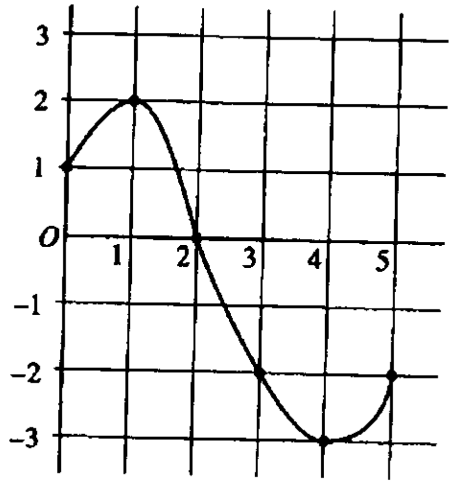
[Viewing Window]
$[-5,5]\times[-3,3]$

(c) Write an equation for each horizontal asymptote of the graph of f.

(d) Find the range of f. Use $f'(x)$ to justify your answer.

Note: $f'(x) = \dfrac{x+2}{(x^2 + x + 1)^{\frac{3}{2}}}$

1995 BC6



Graph of f

Let f be a function whose domain is the closed interval $[0,5]$. The graph of f is shown above.

Let $h(x)=\int_0^{\frac{x}{2}+3} f(t)\,dt$.

(a) Find the domain of h.

(b) Find $h'(2)$.

(c) At what x is $h(x)$ a minimum? Show the analysis that leads to your conclusion.

1998 Calculus BC Scoring Guidelines

2. Let f be the function given by $f(x) = 2xe^{2x}$.
   (a) Find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.
   (b) Find the absolute minimum value of f. Justify that your answer is an absolute minimum.
   (c) What is the range of f?
   (d) Consider the family of functions defined by $y = bxe^{bx}$, where b is a nonzero constant. Show that the absolute minimum value of bxe^{bx} is the same for all nonzero values of b.

(a) $\lim_{x\to-\infty} 2xe^{2x} = 0$

$\lim_{x\to\infty} 2xe^{2x} = \infty$ or DNE

$2\begin{cases} 1\text{: } 0 \text{ as } x \to -\infty \\ 1\text{: } \infty \text{ or DNE as } x \to \infty \end{cases}$

(b) $f'(x) = 2e^{2x} + 2x \cdot 2 \cdot e^{2x} = 2e^{2x}(1+2x) = 0$
if $x = -1/2$

$f(-1/2) = -1/e$ or -0.368 or -0.367

$-1/e$ is an absolute minimum value because:

(i) $f'(x) < 0$ for all $x < -1/2$ and
$f'(x) > 0$ for all $x > -1/2$

–or–

(ii) $f'(x)$ $\quad - \quad | \quad +$
$\qquad\qquad\quad -1/2$

and $x = -1/2$ is the only critical number

3 {
- 1: solves $f'(x) = 0$
- 1: evaluates f at student's critical point
  - 0/1 if not local minimum from student's derivative
- 1: justifies absolute minimum value
  - 0/1 for a local argument
  - 0/1 without explicit symbolic derivative

Note: 0/3 if no absolute minimum based on student's derivative

(c) Range of $f = [-1/e, \infty)$
or $[-0.367, \infty)$
or $[-0.368, \infty)$

1: answer

Note: must include the left-hand endpoint; exclude the right-hand "endpoint"

(d) $y' = be^{bx} + b^2xe^{bx} = be^{bx}(1+bx) = 0$
if $x = -1/b$

At $x = -1/b$, $y = -1/e$

y has an absolute minimum value of $-1/e$ for all nonzero b

3 {
- 1: sets $y' = be^{bx}(1+bx) = 0$
- 1: solves student's $y' = 0$
- 1: evaluates y at a critical number and gets a value independent of b

Note: 0/3 if only considering specific values of b

Question 5

The derivative of a function f is given by $f'(x) = (x-3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$ Therefore, f has a relative minimum at $x = 3$.	$2: \begin{cases} 1: \text{minimum at } x = 3 \\ 1: \text{justification} \end{cases}$
(b) $f''(x) = e^x + (x-3)e^x = (x-2)e^x$ $f''(x) > 0$ for $x > 2$ $f'(x) < 0$ for $0 < x < 3$ Therefore, the graph of f is both decreasing and concave up on the interval $2 < x < 3$.	$3: \begin{cases} 2: f''(x) \\ 1: \text{answer with reason} \end{cases}$
(c) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 7 + \int_1^3 (x-3)e^x\,dx$ $u = x - 3 \quad dv = e^x\,dx$ $du = dx \quad v = e^x$ $f(3) = 7 + (x-3)e^x\Big\vert_1^3 - \int_1^3 e^x\,dx$ $= 7 + \left((x-3)e^x - e^x\right)\Big\vert_1^3$ $= 7 + 3e - e^3$	$4: \begin{cases} 1: \text{uses initial condition} \\ 2: \text{integration by parts} \\ 1: \text{answer} \end{cases}$

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Question 4

Let f be the function defined for $x > 0$, with $f(e) = 2$ and f', the first derivative of f, given by $f'(x) = x^2 \ln x$.

(a) Write an equation for the line tangent to the graph of f at the point $(e, 2)$.

(b) Is the graph of f concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

(c) Use antidifferentiation to find $f(x)$.

(a) $f'(e) = e^2$

An equation for the line tangent to the graph of f at the point $(e, 2)$ is $y - 2 = e^2(x - e)$.

$2 : \begin{cases} 1 : f'(e) \\ 1 : \text{equation of tangent line} \end{cases}$

(b) $f''(x) = x + 2x \ln x.$

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of f is concave up on $(1, 3)$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) Since $f(x) = \int (x^2 \ln x)\, dx$, we consider integration by parts.

$$u = \ln x \qquad dv = x^2\, dx$$
$$du = \frac{1}{x}\, dx \qquad v = \int (x^2)\, dx = \frac{1}{3}x^3$$

Therefore,

$$\begin{aligned} f(x) &= \int (x^2 \ln x)\, dx \\ &= \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3 \cdot \frac{1}{x} \right) dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C. \end{aligned}$$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3} \ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$4 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{uses } f(e) = 2 \\ 1 : \text{answer} \end{cases}$

Question 4

Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of h concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of h at $x = 4$.

(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

(a) $h'(x) = 0$ at $x = \pm\sqrt{2}$

$h'(x)$	$-$	0	$+$	und	$-$	0	$+$
x		$-\sqrt{2}$		0		$\sqrt{2}$	

Local minima at $x = -\sqrt{2}$ and at $x = \sqrt{2}$

$$4 : \begin{cases} 1 : x = \pm\sqrt{2} \\ 1 : \text{analysis} \\ 2 : \text{conclusions} \\ \qquad < -1 > \text{ not dealing with} \\ \qquad\qquad \text{discontinuity at } 0 \end{cases}$$

(b) $h''(x) = 1 + \dfrac{2}{x^2} > 0$ for all $x \neq 0$. Therefore, the graph of h is concave up for all $x \neq 0$.

$$3 : \begin{cases} 1 : h''(x) \\ 1 : h''(x) > 0 \\ 1 : \text{answer} \end{cases}$$

(c) $h'(4) = \dfrac{16 - 2}{4} = \dfrac{7}{2}$

$y + 3 = \dfrac{7}{2}(x - 4)$

1 : tangent line equation

(d) The tangent line is below the graph because the graph of h is concave up for $x > 4$.

1 : answer with reason

1997 AB4

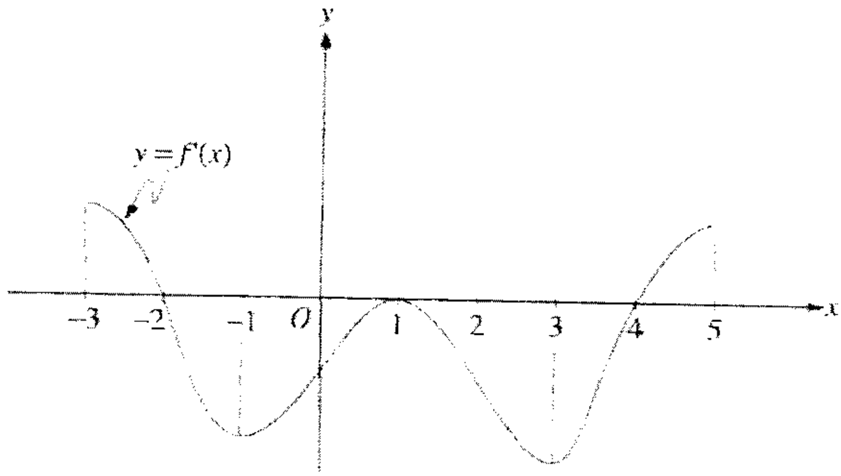
Let f be the function given by $f(x) = x^3 - 6x^2 + p$, where p is an arbitrary constant.

(a) Write an expression for $f'(x)$ and use it to find the relative maximum and minimum values of f in terms of p. Show the analysis that leads to your conclusion.

(b) For what values of the constant p does f have 3 distinct roots?

(c) Find the value of p such that the average value of f over the closed interval $[-1,2]$ is 1.

1996 AB1



Note: This is the graph of the derivative of f, not the graph of f.

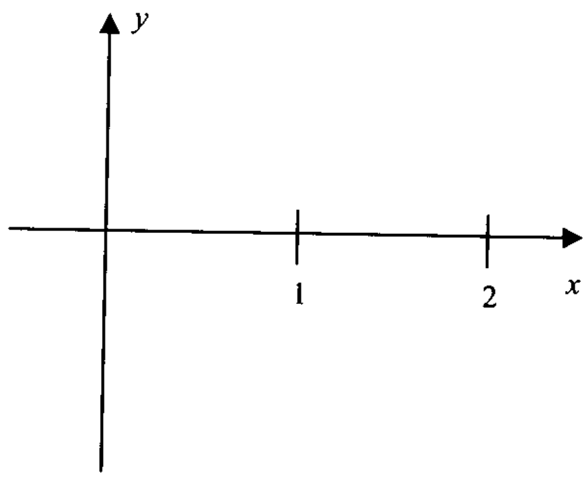
The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-3 < x < 5$.

(a) For what values of x does f have a relative maximum? Why?

(b) For what values of x does f have a relative minimum? Why?

(c) On what intervals is the graph of f concave upward? Use f' to justify your answer.

(d) Suppose that $f(1) = 0$. In the xy-plane provided, draw a sketch that shows the general shape of the graph of the function f on the open interval $0 < x < 2$.



1995 BC5

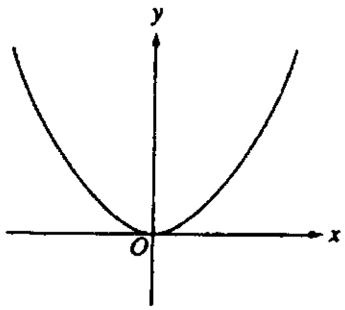


Figure 1
$y = f(x)$

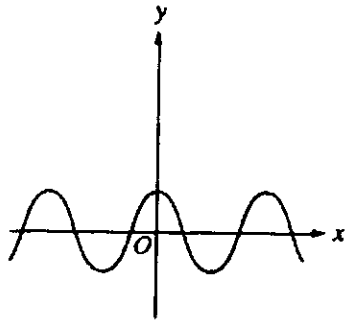


Figure 2
$y = g(x)$

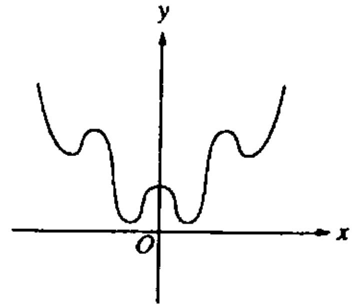
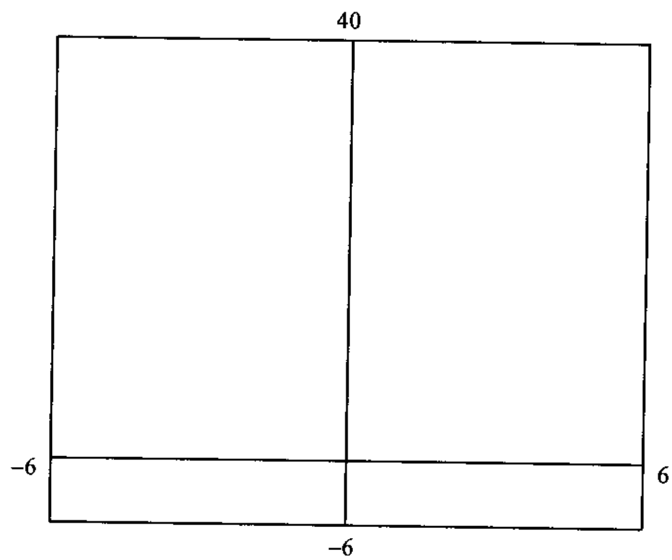


Figure 3

Let $f(x) = x^2$, $g(x) = \cos x$, and $h(x) = x^2 + \cos x$. From the graphs of f and g shown above in Figure 1 and Figure 2, one might think the graph of h should look like the graph in Figure 3.

(a) Sketch the actual graph of h in the viewing window provided below.



Viewing Window
$[-6, 6] \times [-6, 40]$

(b) Use $h''(x)$ to explain why the graph of h does not look like the graph in Figure 3.

(c) Prove that the graph of $y = x^2 + \cos(kx)$ has either no points of inflection or infinitely many points of inflection, depending on the value of the constant k.

1993 AB4/BC3

Let f be the function defined by $f(x) = \ln(2 + \sin x)$ for $\pi \le x \le 2\pi$.

(a) Find the absolute maximum value and the absolute minimum value of f. Show the analysis that leads to your conclusion.

(b) Find the x-coordinate of each inflection point on the graph of f. Justify your answer.

1993 AB1

Let f be the function given by $f(x) = x^3 - 5x^2 + 3x + k$, where k is a constant.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave downward?

(c) Find the value of k for which f has 11 as its relative minimum.

1992 BC4

Let f be a function defined by $f(x) = \begin{cases} 2x - x^2 & \text{for } x \le 1, \\ x^2 + kx + p & \text{for } x > 1. \end{cases}$

(a) For what values of k and p will f be continuous and differentiable at $x = 1$?

(b) For the values of k and p found in part (a), on what interval or intervals is f increasing?

(c) Using the values of k and p found in part (a), find all points of inflection of the graph of f. Support your conclusion.

1992 AB1

Let f be the function defined by $f(x) = 3x^5 - 5x^3 + 2$.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave upward?

(c) Write the equation of each horizontal tangent line to the graph of f.

1991 AB5

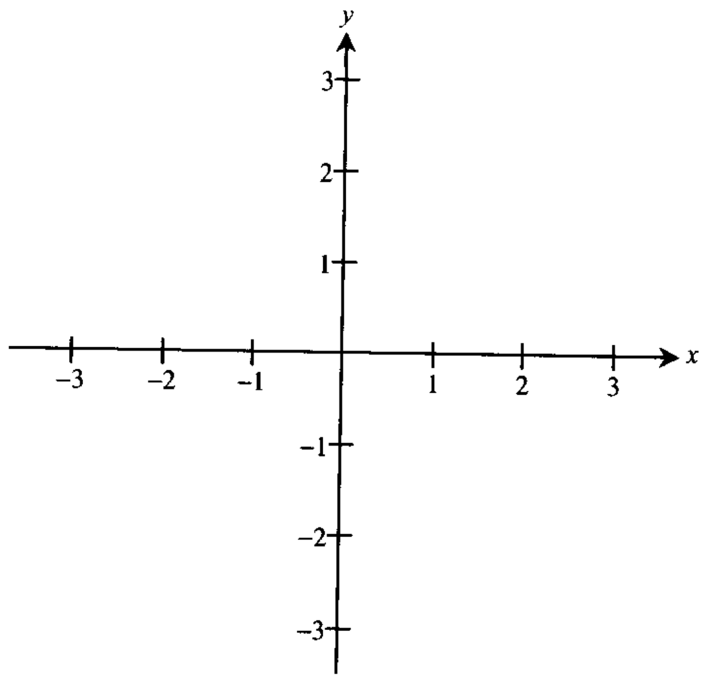
Let f be a function that is <u>even</u> and continuous on the closed interval $[-3,3]$. The function f and its derivatives have the properties indicated in the table below.

x	0	$0<x<1$	1	$1<x<2$	2	$2<x<3$
$f(x)$	1	Positive	0	Negative	-1	Negative
$f'(x)$	Undefined	Negative	0	Negative	Undefined	Positive
$f''(x)$	Undefined	Positive	0	Negative	Undefined	Negative

(a) Find the x-coordinate of each point at which f attains an absolute maximum value or an absolute minimum value. For each x-coordinate you give, state whether f attains an absolute maximum or an absolute minimum.

(b) Find the x-coordinate of each point of inflection on the graph of f. Justify your answer.

(c) In the xy-plane provided below, sketch the graph of a function with all the given characteristics of f.



1990 AB5

Let f be the function defined by $f(x) = \sin^2 x - \sin x$ for $0 \le x \le \frac{3\pi}{2}$.

(a) Find the x-intercepts of the graph of f.

(b) Find the intervals on which f is increasing.

(c) Find the absolute maximum value and the absolute minimum value of f. Justify your answer.

1989 BC3

Consider the function f defined by $f(x) = e^x \cos x$ with domain $[0, 2\pi]$.

(a) Find the absolute maximum and minimum values of $f(x)$.

(b) Find the intervals on which f is increasing.

(c) Find the x-coordinate of each point of inflection of the graph of f.

1991 BC2

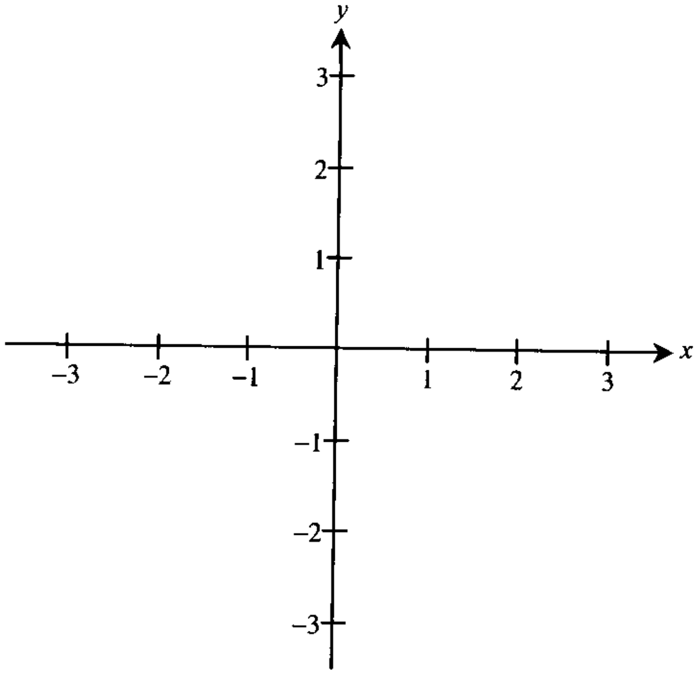
Let f be the function defined by $f(x) = xe^{1-x}$ for all real numbers x.

(a) Find each interval on which f is increasing.

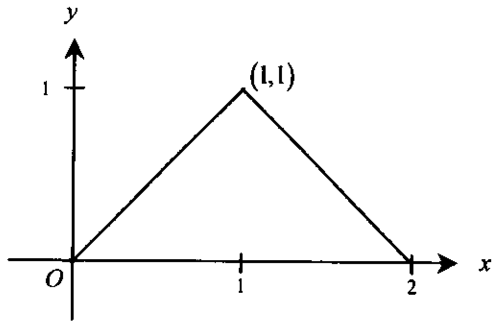
(b) Find the range of f.

(c) Find the x-coordinate of each point of inflection of the graph of f.

(d) Using the results found in parts (a), (b), and (c), sketch the graph of f in the xy-plane provided below. (Indicate all intercepts.)



1993 AB5



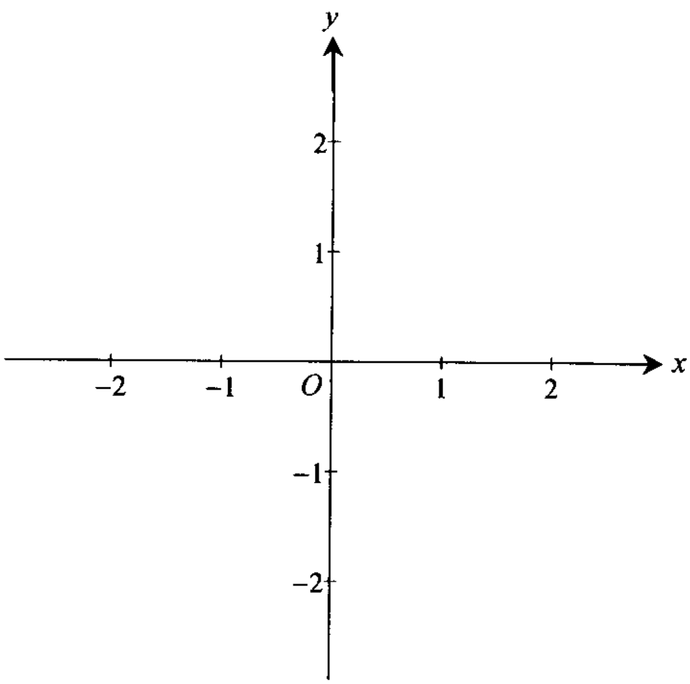
Note: This is the graph of the derivative of f, not the graph of f.

The figure above shows the graph of f', the derivative of f. The domain of f is the set of all x such that $0 < x < 2$.

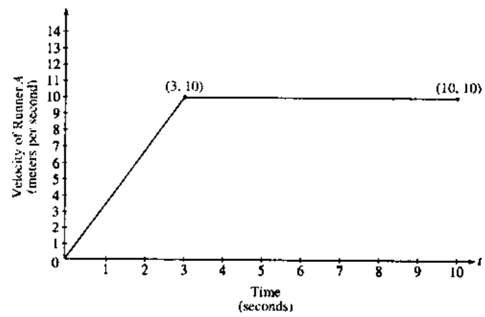
(a) Write an expression for $f'(x)$ in terms of x .

(b) Given that $f(1) = 0$, write an expression for $f(x)$ in terms of x .

(c) In the xy-plane provided below, sketch the graph of $y = f(x)$.



AP Calculus AB–2 / BC–2 — 2000



Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \dfrac{24t}{2t+3}$.

(a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.

(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

(c) Find the total distance run by Runner A and the total distance run by Runner B over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

(a) Runner A: velocity $= \dfrac{10}{3} \cdot 2 = \dfrac{20}{3}$

$= 6.666$ or 6.667 meters/sec

Runner B: $v(2) = \dfrac{48}{7} = 6.857$ meters/sec

$$2 \begin{cases} 1: \text{ velocity for Runner } A \\ 1: \text{ velocity for Runner } B \end{cases}$$

(b) Runner A: acceleration $= \dfrac{10}{3} = 3.333$ meters/sec^2

Runner B: $a(2) = v'(2) = \left.\dfrac{72}{(2t+3)^2}\right|_{t=2}$

$= \dfrac{72}{49} = 1.469$ meters/sec^2

$$2 \begin{cases} 1: \text{ acceleration for Runner } A \\ 1: \text{ acceleration for Runner } B \end{cases}$$

(c) Runner A: distance $= \dfrac{1}{2}(3)(10) + 7(10) = 85$ meters

Runner B: distance $= \displaystyle\int_0^{10} \frac{24t}{2t+3}\,dt = 83.336$ meters

$$4 \begin{cases} 2: \text{ distance for Runner } A \\ \quad 1: \text{ method} \\ \quad 1: \text{ answer} \\ 2: \text{ distance for Runner } B \\ \quad 1: \text{ integral} \\ \quad 1: \text{ answer} \end{cases}$$

(units) meters/sec in part (a), meters/sec^2 in part (b), and meters in part (c), or equivalent.

1: units

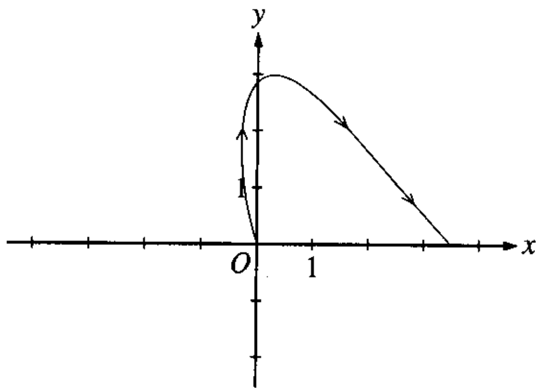
1. A particle moves in the xy–plane so that its position at any time t, $0 \le t \le \pi$, is given by $x(t) = \dfrac{t^2}{2} - \ln(1+t)$ and $y(t) = 3\sin t$.

   (a) Sketch the path of the particle in the xy–plane below. Indicate the direction of motion along the path.

   (b) At what time t, $0 \le t \le \pi$, does $x(t)$ attain its minimum value? What is the position $(x(t), y(t))$ of the particle at this time?

   (c) At what time t, $0 < t < \pi$, is the particle on the y–axis? Find the speed and the acceleration vector of the particle at this time.

(a)



$2\begin{cases} 1\text{: graph} \\ 1\text{: direction} \end{cases}$

(b) $x'(t) = t - \dfrac{1}{1+t} = 0$

$t^2 + t - 1 = 0$

$t = \dfrac{-1+\sqrt{5}}{2}$ or $t = 0.618$ in $[0, \pi]$

$x(0.618) = -0.290 \qquad y(0.618) = 1.738$

$3\begin{cases} 1\text{: } x'(t) = 0 \\ 1\text{: solution for } t \\ 1\text{: position} \end{cases}$

(c) $x(t) = \dfrac{t^2}{2} - \ln(1+t) = 0$

$t = 1.285$ or 1.286

$x'(t) = t - \dfrac{1}{1+t} \qquad y'(t) = 3\cos t$

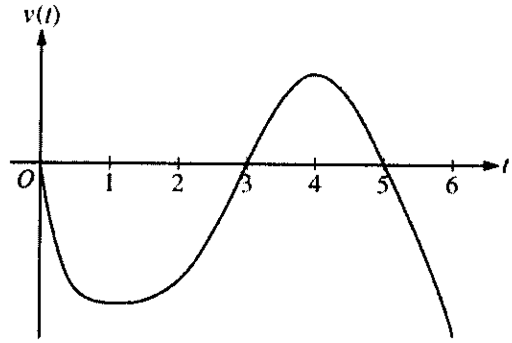
speed $= \sqrt{(x'(1.286))^2 + (y'(1.286))^2} = 1.196$

$x''(t) = 1 + \dfrac{1}{(1+t)^2} \qquad y''(t) = -3\sin t$

acceleration vector $= \langle x''(1.286), y''(1.286) \rangle$
$= \langle 1.191, -2.879 \rangle$

$4\begin{cases} 1\text{: } x(t) = 0 \\ 1\text{: solution for } t \\ 1\text{: speed} \\ 1\text{: acceleration vector} \end{cases}$

Question 4



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

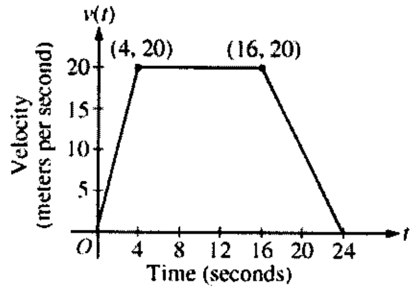
(b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

(a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$. $x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$ $x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$ Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$.	3 : { 1 : identifies $t = 3$ as a candidate 1 : considers $\int_0^6 v(t)\,dt$ 1 : conclusion
(b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$. By the Intermediate Value Theorem, there are three values of t for which the particle is at $x(t) = -8$.	3 : { 1 : positions at $t = 3$, $t = 5$, and $t = 6$ 1 : description of motion 1 : conclusion
(c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.	1 : answer with reason
(d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.	2 : { 1 : answer 1 : justification

Question 5



A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.

(a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

(b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

(c) Let $a(t)$ be the car's acceleration at time t, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

(d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

(a) $\int_0^{24} v(t)\,dt = \frac{1}{2}(4)(20) + (12)(20) + \frac{1}{2}(8)(20) = 360$

The car travels 360 meters in these 24 seconds.

$$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{meaning with units} \end{cases}$$

(b) $v'(4)$ does not exist because

$$\lim_{t \to 4^-} \left(\frac{v(t) - v(4)}{t - 4} \right) = 5 \ne 0 = \lim_{t \to 4^+} \left(\frac{v(t) - v(4)}{t - 4} \right).$$

$$v'(20) = \frac{20 - 0}{16 - 24} = -\frac{5}{2} \ \text{m/sec}^2$$

$$3 : \begin{cases} 1 : v'(4) \text{ does not exist, with explanation} \\ 1 : v'(20) \\ 1 : \text{units} \end{cases}$$

(c)

$$a(t) = \begin{cases} 5 & \text{if } 0 < t < 4 \\ 0 & \text{if } 4 < t < 16 \\ -\frac{5}{2} & \text{if } 16 < t < 24 \end{cases}$$

$a(t)$ does not exist at $t = 4$ and $t = 16$.

$$2 : \begin{cases} 1 : \text{finds the values } 5,\ 0,\ -\frac{5}{2} \\ 1 : \text{identifies constants with correct intervals} \end{cases}$$

(d) The average rate of change of v on $[8, 20]$ is

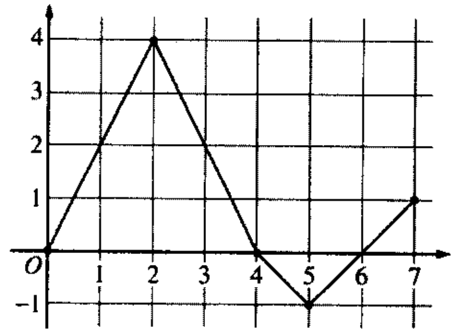
$$\frac{v(20) - v(8)}{20 - 8} = -\frac{5}{6} \ \text{m/sec}^2.$$

No, the Mean Value Theorem does not apply to v on $[8, 20]$ because v is not differentiable at $t = 16$.

$$2 : \begin{cases} 1 : \text{average rate of change of } v \text{ on } [8, 20] \\ 1 : \text{answer with explanation} \end{cases}$$

Question 5

Let f be a function defined on the closed interval $[0,7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.



Graph of f

(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Explain your reasoning.

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.

(a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(4+2) = 3$

$g'(3) = f(3) = 2$

$g''(3) = f'(3) = \frac{0-4}{4-2} = -2$

$3 : \begin{cases} 1 : g(3) \\ 1 : g'(3) \\ 1 : g''(3) \end{cases}$

(b) $\frac{g(3)-g(0)}{3} = \frac{1}{3}\int_0^3 f(t)\,dt$

$= \frac{1}{3}\left(\frac{1}{2}(2)(4) + \frac{1}{2}(4+2)\right) = \frac{7}{3}$

$2 : \begin{cases} 1 : g(3) - g(0) = \int_0^3 f(t)\,dt \\ 1 : \text{answer} \end{cases}$

(c) There are two values of c.

We need $\frac{7}{3} = g'(c) = f(c)$

The graph of f intersects the line $y = \frac{7}{3}$ at two places between 0 and 3.

$2 : \begin{cases} 1 : \text{answer of 2} \\ 1 : \text{reason} \end{cases}$

Note: 1/2 if answer is 1 by MVT

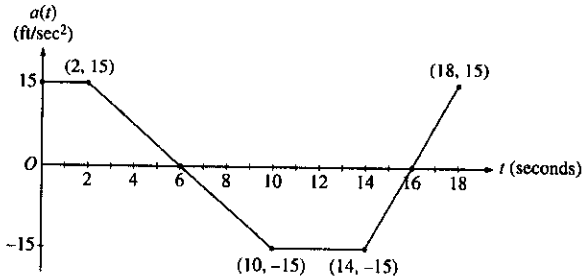
(d) $x = 2$ and $x = 5$

because $g' = f$ changes from increasing to decreasing at $x = 2$, and from decreasing to increasing at $x = 5$.

$2 : \begin{cases} 1 : x = 2 \text{ and } x = 5 \text{ only} \\ 1 : \text{justification} \\ \quad \text{(ignore discussion at } x = 4) \end{cases}$

Question 3

A car is traveling on a straight road with velocity 55 ft/sec at time $t = 0$. For $0 \le t \le 18$ seconds, the car's acceleration $a(t)$, in ft/sec^2, is the piecewise linear function defined by the graph above.



(a) Is the velocity of the car increasing at $t = 2$ seconds? Why or why not?

(b) At what time in the interval $0 \le t \le 18$, other than $t = 0$, is the velocity of the car 55 ft/sec? Why?

(c) On the time interval $0 \le t \le 18$, what is the car's absolute maximum velocity, in ft/sec, and at what time does it occur? Justify your answer.

(d) At what times in the interval $0 \le t \le 18$, if any, is the car's velocity equal to zero? Justify your answer.

(a) Since $v'(2) = a(2)$ and $a(2) = 15 > 0$, the velocity is increasing at $t = 2$.

1 : answer and reason

(b) At time $t = 12$ because

$$v(12) - v(0) = \int_0^{12} a(t)\,dt = 0.$$

$$2 : \begin{cases} 1 : t = 12 \\ 1 : \text{reason} \end{cases}$$

(c) The absolute maximum velocity is 115 ft/sec at $t = 6$.

The absolute maximum must occur at $t = 6$ or at an endpoint.

$$v(6) = 55 + \int_0^6 a(t)\,dt$$
$$= 55 + 2(15) + \frac{1}{2}(4)(15) = 115 > v(0)$$
$$\int_6^{18} a(t)\,dt < 0 \text{ so } v(18) < v(6)$$

$$4 : \begin{cases} 1 : t = 6 \\ 1 : \text{absolute maximum velocity} \\ 1 : \text{identifies } t = 6 \text{ and} \\ \quad t = 18 \text{ as candidates} \\ \quad \text{or} \\ \quad \text{indicates that } v \text{ increases,} \\ \quad \text{decreases, then increases} \\ 1 : \text{eliminates } t = 18 \end{cases}$$

(d) The car's velocity is never equal to 0. The absolute minimum occurs at $t = 16$ where

$$v(16) = 115 + \int_6^{16} a(t)\,dt = 115 - 105 = 10 > 0.$$

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$$

3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24–hour period.

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

(a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t)\,dt$. Using correct units, explain the meaning of your answer in terms of water flow.

(b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

(c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$. Use $Q(t)$ to approximate the average rate of water flow during the 24–hour time period. Indicate units of measure.

(a) $$\int_0^{24} R(t)\,dt \approx 6[R(3) + R(9) + R(15) + R(21)]$$
$$= 6[10.4 + 11.2 + 11.3 + 10.2]$$
$$= 258.6 \text{ gallons}$$

This is an approximation to the total flow in gallons of water from the pipe in the 24–hour period.

$3 \begin{cases} 1\text{: } R(3) + R(9) + R(15) + R(21) \\ 1\text{: answer} \\ 1\text{: explanation} \end{cases}$

(b) Yes;

Since $R(0) = R(24) = 9.6$, the Mean Value Theorem guarantees that there is a t, $0 < t < 24$, such that $R'(t) = 0$.

$2 \begin{cases} 1\text{: answer} \\ 1\text{: MVT or equivalent} \end{cases}$

(c) Average rate of flow

$\approx$ average value of $Q(t)$

$$= \frac{1}{24}\int_0^{24} \frac{1}{79}(768 + 23t - t^2)\,dt$$

$$= 10.785 \text{ gal/hr} \quad \text{or} \quad 10.784 \text{ gal/hr}$$

$3 \begin{cases} 1\text{: limits and average value constant} \\ 1\text{: } Q(t) \text{ as integrand} \\ 1\text{: answer} \end{cases}$

(units) Gallons in part (a) and gallons/hr in part (c), or equivalent.

1: units

Question 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

(a) $L'(5.5) \approx \dfrac{L(7) - L(4)}{7 - 4} = \dfrac{150 - 126}{3} = 8$ people per hour

$2: \begin{cases} 1: \text{estimate} \\ 1: \text{units} \end{cases}$

(b) The average number of people waiting in line during the first 4 hours is approximately

$$\frac{1}{4}\left(\frac{L(0) + L(1)}{2}(1 - 0) + \frac{L(1) + L(3)}{2}(3 - 1) + \frac{L(3) + L(4)}{2}(4 - 3)\right)$$

$= 155.25$ people

$2: \begin{cases} 1: \text{trapezoidal sum} \\ 1: \text{answer} \end{cases}$

(c) L is differentiable on $[0, 9]$ so the Mean Value Theorem implies $L'(t) > 0$ for some t in $(1, 3)$ and some t in $(4, 7)$. Similarly, $L'(t) < 0$ for some t in $(3, 4)$ and some t in $(7, 8)$. Then, since L' is continuous on $[0, 9]$, the Intermediate Value Theorem implies that $L'(t) = 0$ for at least three values of t in $[0, 9]$.

$3: \begin{cases} 1: \text{considers change in sign of } L' \\ 1: \text{analysis} \\ 1: \text{conclusion} \end{cases}$

OR

The continuity of L on $[1, 4]$ implies that L attains a maximum value there. Since $L(3) > L(1)$ and $L(3) > L(4)$, this maximum occurs on $(1, 4)$. Similarly, L attains a minimum on $(3, 7)$ and a maximum on $(4, 8)$. L is differentiable, so $L'(t) = 0$ at each relative extreme point on $(0, 9)$. Therefore $L'(t) = 0$ for at least three values of t in $[0, 9]$.

OR

$3: \begin{cases} 1: \text{considers relative extrema of } L \text{ on } (0, 9) \\ 1: \text{analysis} \\ 1: \text{conclusion} \end{cases}$

[*Note: There is a function* L *that satisfies the given conditions with* $L'(t) = 0$ *for exactly three values of* t.]

(d) $\int_0^3 r(t)\, dt = 972.784$

There were approximately 973 tickets sold by 3 P.M.

$2: \begin{cases} 1: \text{integrand} \\ 1: \text{limits and answer} \end{cases}$

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Question 5

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

Solution	Points
(a) $r(5.4) \approx r(5) + r'(5)\Delta t = 30 + 2(0.4) = 30.8$ ft Since the graph of r is concave down on the interval $5 < t < 5.4$, this estimate is greater than $r(5.4)$.	$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{conclusion with reason} \end{cases}$
(b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$ $\left.\frac{dV}{dt}\right\|_{t=5} = 4\pi(30)^2 2 = 7200\pi \text{ ft}^3/\text{min}$	$3 : \begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$
(c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$ $= 19.3$ ft $\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.	$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{explanation} \end{cases}$
(d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$.	1 : conclusion with reason
Units of ft^3/min in part (b) and ft in part (c)	1 : units in (b) and (c)

Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

(a) Average acceleration of rocket A is $\dfrac{v(80)-v(0)}{80-0} = \dfrac{49-5}{80} = \dfrac{11}{20}\ \text{ft}/\text{sec}^2$	1 : answer
(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds. A midpoint Riemann sum is $20[v(20)+v(40)+v(60)]$ $= 20[22+35+44] = 2020$ ft	$3 : \begin{cases} 1 : \text{explanation} \\ 1 : \text{uses } v(20), v(40), v(60) \\ 1 : \text{value} \end{cases}$
(c) Let $v_B(t)$ be the velocity of rocket B at time t. $v_B(t) = \int \dfrac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$ $2 = v_B(0) = 6 + C$ $v_B(t) = 6\sqrt{t+1} - 4$ $v_B(80) = 50 > 49 = v(80)$ Rocket B is traveling faster at time $t = 80$ seconds.	$4 : \begin{cases} 1 : 6\sqrt{t+1} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{finds } v_B(80), \text{ compares to } v(80), \\ \quad \text{and draws a conclusion} \end{cases}$
Units of ft/sec^2 in (a) and ft in (b)	1 : units in (a) and (b)

Question 3

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

(a) $\dfrac{T(8)-T(6)}{8-6} = \dfrac{55-62}{2} = -\dfrac{7}{2}\,°\text{C/cm}$

1 : answer

(b) $\dfrac{1}{8}\int_0^8 T(x)\,dx$

Trapezoidal approximation for $\int_0^8 T(x)\,dx$:

$A = \dfrac{100+93}{2}\cdot 1 + \dfrac{93+70}{2}\cdot 4 + \dfrac{70+62}{2}\cdot 1 + \dfrac{62+55}{2}\cdot 2$

Average temperature $\approx \dfrac{1}{8}A = 75.6875°\text{C}$

3 : $\begin{cases} 1 : \frac{1}{8}\int_0^8 T(x)\,dx \\ 1 : \text{trapezoidal sum} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45°\text{C}$

The temperature drops 45°C from the heated end of the wire to the other end of the wire.

2 : $\begin{cases} 1 : \text{value} \\ 1 : \text{meaning} \end{cases}$

(d) Average rate of change of temperature on $[1, 5]$ is $\dfrac{70-93}{5-1} = -5.75$.

Average rate of change of temperature on $[5, 6]$ is $\dfrac{62-70}{6-5} = -8$.

No. By the MVT, $T'(c_1) = -5.75$ for some c_1 in the interval $(1, 5)$ and $T'(c_2) = -8$ for some c_2 in the interval $(5, 6)$. It follows that T' must decrease somewhere in the interval (c_1, c_2). Therefore T'' is not positive for every x in $[0, 8]$.

2 : $\begin{cases} 1 : \text{two slopes of secant lines} \\ 1 : \text{answer with explanation} \end{cases}$

Units of °C/cm in (a), and °C in (b) and (c)

1 : units in (a), (b), and (c)

Question 3

A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points along the length of the blood vessel, where x represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

Distance x (mm)	0	60	120	180	240	300	360
Diameter $B(x)$ (mm)	24	30	28	30	26	24	26

(a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.

(b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.

(c) Using correct units, explain the meaning of $\pi \int_{125}^{275} \left(\frac{B(x)}{2} \right)^2 dx$ in terms of the blood vessel.

(d) Explain why there must be at least one value x, for $0 < x < 360$, such that $B''(x) = 0$.

(a) $\frac{1}{360} \int_0^{360} \frac{B(x)}{2}\, dx$

$2 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

(b) $\frac{1}{360}\left[120\left(\frac{B(60)}{2} + \frac{B(180)}{2} + \frac{B(300)}{2}\right)\right] =$

$\frac{1}{360}[60(30 + 30 + 24)] = 14$

$2 : \begin{cases} 1 : B(60) + B(180) + B(300) \\ 1 : \text{answer} \end{cases}$

(c) $\frac{B(x)}{2}$ is the radius, so $\pi\left(\frac{B(x)}{2}\right)^2$ is the area of the cross section at x. The expression is the volume in mm^3 of the blood vessel between 125 mm and 275 mm from the end of the vessel.

$2 : \begin{cases} 1 : \text{volume in mm}^3 \\ 1 : \text{between } x = 125 \text{ and } x = 275 \end{cases}$

(d) By the MVT, $B'(c_1) = 0$ for some c_1 in $(60, 180)$ and $B'(c_2) = 0$ for some c_2 in $(240, 360)$. The MVT applied to $B'(x)$ shows that $B''(x) = 0$ for some x in the interval (c_1, c_2).

$3 : \begin{cases} 2 : \text{explains why there are two values of } x \text{ where } B'(x) \text{ has the same value} \\ 1 : \text{explains why that means } B''(x) = 0 \text{ for } 0 < x < 360 \end{cases}$

Note: max 1/3 if only explains why $B'(x) = 0$ at some x in $(0, 360)$.

Question 2

The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

t (days)	$W(t)$ (°C)
0	20
3	31
6	28
9	24
12	22
15	21

(a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

(c) A student proposes the function P, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

(d) Use the function P defined in part (c) to find the average value, in degrees Celsius, of $P(t)$ over the time interval $0 \le t \le 15$ days.

(a) Difference quotient; e.g.

$$W'(12) \approx \frac{W(15) - W(12)}{15 - 12} = -\frac{1}{3} \text{ °C/day or}$$

$$W'(12) \approx \frac{W(12) - W(9)}{12 - 9} = -\frac{2}{3} \text{ °C/day or}$$

$$W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = -\frac{1}{2} \text{ °C/day}$$

2 : { 1 : difference quotient; 1 : answer (with units) }

(b) $\frac{3}{2}(20 + 2(31) + 2(28) + 2(24) + 2(22) + 21) = 376.5$

Average temperature $\approx \frac{1}{15}(376.5) = 25.1$ °C

2 : { 1 : trapezoidal method; 1 : answer }

(c) $P'(12) = 10e^{-t/3} - \frac{10}{3}te^{-t/3}\Big|_{t=12}$

$= -30e^{-4} = -0.549$ °C/day

This means that the temperature is decreasing at the rate of 0.549 °C/day when $t = 12$ days.

2 : { 1 : $P'(12)$ (with or without units); 1 : interpretation }

(d) $\frac{1}{15}\int_0^{15}\left(20 + 10te^{-t/3}\right)dt = 25.757$ °C

3 : { 1 : integrand; 1 : limits and average value constant; 1 : answer }

1990 BC6

Let f and g be continuous functions with the following properties.

(i) $g(x) = A - f(x)$ where A is a constant

(ii) $\int_1^2 f(x)\,dx = \int_2^3 g(x)\,dx$

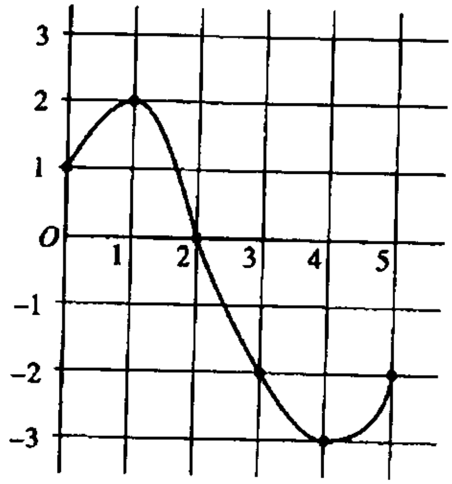
(iii) $\int_2^3 f(x)\,dx = -3A$

(a) Find $\int_1^3 f(x)\,dx$ in terms of A.

(b) Find the average value of $g(x)$ in terms of A, over the interval $[1,3]$.

(c) Find the value of k if $\int_0^1 f(x+1)\,dx = kA$.

1995 BC6



Graph of f

Let f be a function whose domain is the closed interval $[0,5]$. The graph of f is shown above.

Let $h(x) = \int_0^{\frac{x}{2}+3} f(t)\,dt$.

(a) Find the domain of h.

(b) Find $h'(2)$.

(c) At what x is $h(x)$ a minimum? Show the analysis that leads to your conclusion.

1991 BC4

Let $F(x) = \int_1^{2x} \sqrt{t^2 + t}\, dt$.

(a) Find $F'(x)$.

(b) Find the domain of F.

(c) Find $\lim_{x \to \frac{1}{2}} F(x)$.

(d) Find the length of the curve $y = F(x)$ for $1 \le x \le 2$.

1994 AB 6

Let $F(x)=\int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.

(a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0,1]$ to approximate $F(1)$.

(b) On what intervals is F increasing?

(c) If the average rate of change of F on the closed interval $[1,3]$ is k, find $\int_1^3 \sin(t^2)\,dt$ in terms of k.

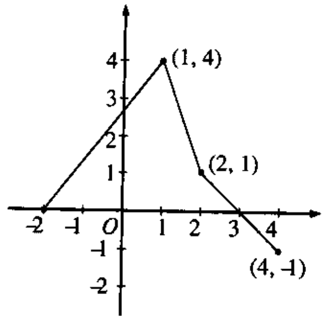
5. The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)\,dt$.

   (a) Compute $g(4)$ and $g(-2)$.

   (b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

   (c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

   (d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x–coordinates of points of inflection of the graph of g? Justify your answer.



(a) $g(4) = \int_1^4 f(t)\,dt = \frac{3}{2} + 1 + \frac{1}{2} - \frac{1}{2} = \frac{5}{2}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\frac{1}{2}(12) = -6$

$2\begin{cases} 1\text{: } g(4) \\ 1\text{: } g(-2) \end{cases}$

(b) $g'(1) = f(1) = 4$

1: answer

(c) g is increasing on $[-2, 3]$ and decreasing on $[3, 4]$.

Therefore, g has absolute minimum at an endpoint of $[-2, 4]$.

Since $g(-2) = -6$ and $g(4) = \frac{5}{2}$,

the absolute minimum value is -6.

$3\begin{cases} 1\text{: interior analysis} \\ 1\text{: endpoint analysis} \\ 1\text{: answer} \end{cases}$

(d) One; $x = 1$

On $(-2, 1)$, $g''(x) = f'(x) > 0$

On $(1, 2)$, $g''(x) = f'(x) < 0$

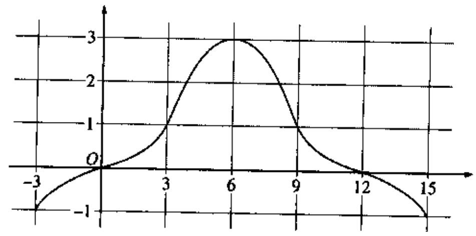
On $(2, 4)$, $g''(x) = f'(x) < 0$

Therefore $(1, g(1))$ is a point of inflection and $(2, g(2))$ is not.

$3\begin{cases} 1\text{: choice of } x = 1 \text{ only} \\ 1\text{: show } (1, g(1)) \text{ is a point of inflection} \\ 1\text{: show } (2, g(2)) \text{ is not a point of inflection} \end{cases}$

Question 4

The graph of a differentiable function f on the closed interval $[-3, 15]$ is shown in the figure above. The graph of f has a horizontal tangent line at $x = 6$. Let $g(x) = 5 + \int_6^x f(t)\,dt$ for $-3 \le x \le 15$.



Graph of f

(a) Find $g(6)$, $g'(6)$, and $g''(6)$.

(b) On what intervals is g decreasing? Justify your answer.

(c) On what intervals is the graph of g concave down? Justify your answer.

(d) Find a trapezoidal approximation of $\int_{-3}^{15} f(t)\,dt$ using six subintervals of length $\Delta t = 3$.

(a) $g(6) = 5 + \int_6^6 f(t)\,dt = 5$

$g'(6) = f(6) = 3$

$g''(6) = f'(6) = 0$

$3\begin{cases} 1: g(6) \\ 1: g'(6) \\ 1: g''(6) \end{cases}$

(b) g is decreasing on $[-3, 0]$ and $[12, 15]$ since $g'(x) = f(x) < 0$ for $x < 0$ and $x > 12$.

$3\begin{cases} 1: [-3,0] \\ 1: [12,15] \\ 1: \text{justification} \end{cases}$

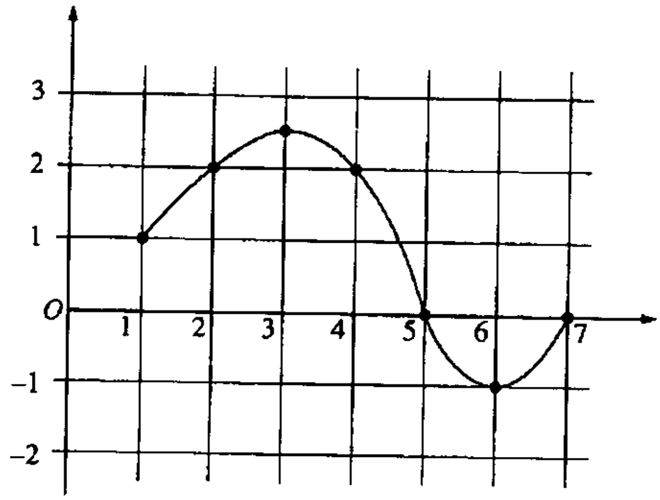
(c) The graph of g is concave down on $(6, 15)$ since $g' = f$ is decreasing on this interval.

$2\begin{cases} 1: \text{interval} \\ 1: \text{justification} \end{cases}$

(d) $\frac{3}{2}(-1 + 2(0 + 1 + 3 + 1 + 0) - 1)$
$= 12$

1 : trapezoidal method

1995 AB6



The graph of a differentiable function f on the closed interval $[1, 7]$ is shown above.
Let $h(x) = \int_1^x f(t)\,dt$ for $1 \le x \le 7$.

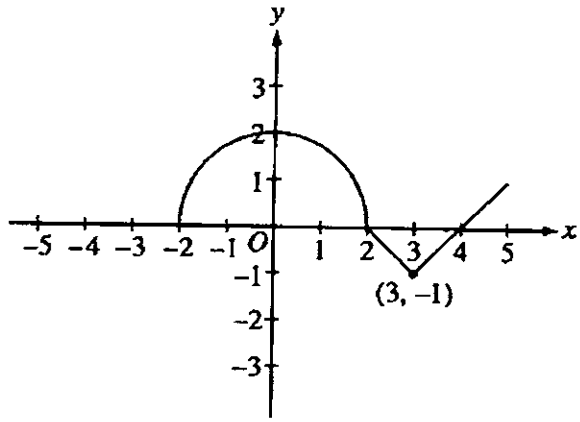
(a) Find $h(1)$.

(b) Find $h'(4)$.

(c) On what interval or intervals is the graph of h concave upward? Justify your answer.

(d) Find the value of x at which h has its minimum on the closed interval $[1, 7]$. Justify your answer.

1997 AB5/BC5



The graph of the function f consists of a semicircle and two line segments as shown above. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(3)$.

(b) Find all the values of x on the open interval $(-2, 5)$ at which g has a relative maximum. Justify your answer.

(c) Write an equation for the line tangent to the graph of g at $x = 3$.

(d) Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-2, 5)$. Justify your answer.

1998 Calculus BC Scoring Guidelines

5. The temperature outside a house during a 24-hour period is given by

$$F(t) = 80 - 10\cos\left(\frac{\pi t}{12}\right),\ 0 \le t \le 24,$$

where $F(t)$ is measured in degrees Fahrenheit and t is measured in hours.

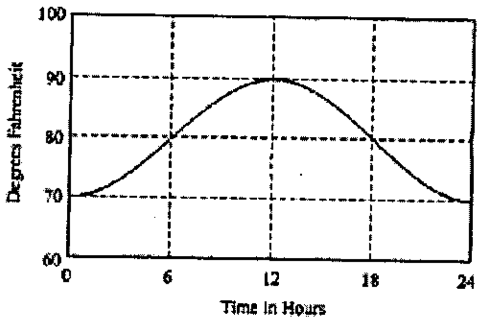
(a) Sketch the graph of F on the grid below.

(b) Find the average temperature, to the nearest degree Fahrenheit, between $t = 6$ and $t = 14$.

(c) An air conditioner cooled the house whenever the outside temperature was at or above 78 degrees Fahrenheit. For what values of t was the air conditioner cooling the house?

(d) The cost of cooling the house accumulates at the rate of \$0.05 per hour for each degree the outside temperature exceeds 78 degrees Fahrenheit. What was the total cost, to the nearest cent, to cool the house for this 24-hour period?

(a)



1: bell-shaped graph
minimum 70 at $t = 0$, $t = 24$ only
maximum 90 at $t = 12$ only

(b) $$\text{Avg.} = \frac{1}{14-6}\int_6^{14}\left[80 - 10\cos\left(\frac{\pi t}{12}\right)\right]dt$$

$$= \frac{1}{8}(697.2957795)$$

$$= 87.162 \text{ or } 87.161$$

$$\approx 87^\circ\ \text{F}$$

3 $\begin{cases} 2: \text{ integral} \\ \quad 1: \text{ limits and } 1/(14-6) \\ \quad 1: \text{ integrand} \\ 1: \text{ answer} \\ \quad 0/1 \text{ if integral not of the form} \\ \quad \dfrac{1}{b-a}\displaystyle\int_a^b F(t)\,dt \end{cases}$

(c) $$\left[80 - 10\cos\left(\frac{\pi t}{12}\right)\right] - 78 \ge 0$$

$$2 - 10\cos\left(\frac{\pi t}{12}\right) \ge 0$$

$$\left.\begin{matrix} 5.230 \\ \text{or} \\ 5.231 \end{matrix}\right\} \le t \le \left\{\begin{matrix} 18.769 \\ \text{or} \\ 18.770 \end{matrix}\right.$$

2 $\begin{cases} 1: \text{ inequality or equation} \\ 1: \text{ solutions with interval} \end{cases}$

(d) $$C = 0.05\int_{5.231 \text{ or } 5.230}^{18.770 \text{ or } 18.769}\left(\left[80 - 10\cos\left(\frac{\pi t}{12}\right)\right] - 78\right)dt$$

$$= 0.05(101.92741) = 5.096 \approx \$5.10$$

3 $\begin{cases} 2: \text{ integral} \\ \quad 1: \text{ limits and } 0.05 \\ \quad 1: \text{ integrand} \\ 1: \text{ answer} \\ \quad 0/1 \text{ if integral not of the form} \\ \quad k\displaystyle\int_a^b (F(t) - 78)\,dt \end{cases}$

AP® CALCULUS BC
2007 SCORING GUIDELINES

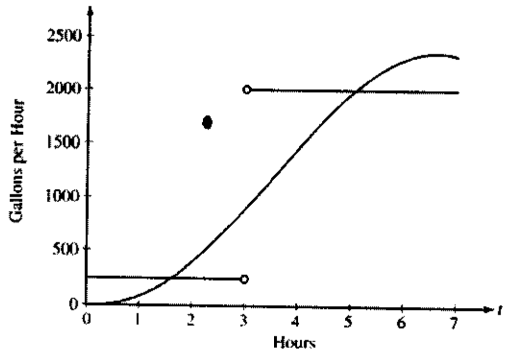
Question 2

The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where t is measured in hours. In this model, rates are given as follows:

(i) The rate at which water enters the tank is $f(t) = 100t^2 \sin(\sqrt{t})$ gallons per hour for $0 \le t \le 7$.

(ii) The rate at which water leaves the tank is

$$g(t) = \begin{cases} 250 & \text{for } 0 \le t < 3 \\ 2000 & \text{for } 3 < t \le 7 \end{cases} \text{ gallons per hour.}$$



The graphs of f and g, which intersect at $t = 1.617$ and $t = 5.076$, are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

(a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

(b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for each answer.

(c) For $0 \le t \le 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

(a) $\int_0^7 f(t)\,dt \approx 8264$ gallons

$$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$$

(b) The amount of water in the tank is decreasing on the intervals $0 \le t \le 1.617$ and $3 \le t \le 5.076$ because $f(t) < g(t)$ for $0 \le t < 1.617$ and $3 < t < 5.076$.

$$2: \begin{cases} 1: \text{intervals} \\ 1: \text{reason} \end{cases}$$

(c) Since $f(t) - g(t)$ changes sign from positive to negative only at $t = 3$, the candidates for the absolute maximum are at $t = 0,\ 3,$ and 7.

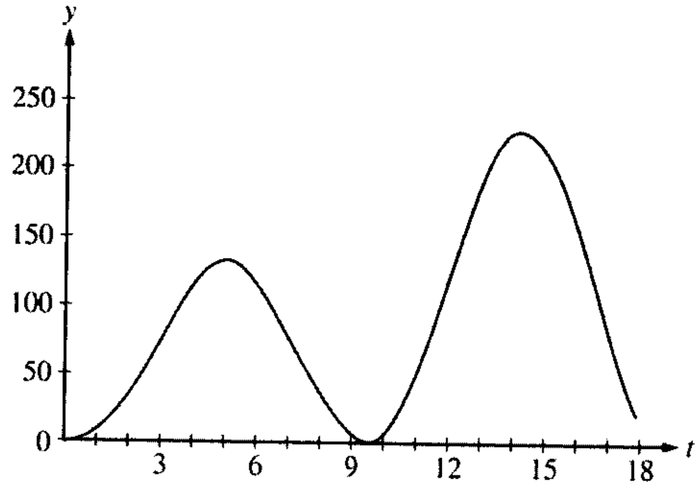
t (hours)	gallons of water
0	5000
3	$5000 + \int_0^3 f(t)\,dt - 250(3) = 5126.591$
7	$5126.591 + \int_3^7 f(t)\,dt - 2000(4) = 4513.807$

The amount of water in the tank is greatest at 3 hours. At that time, the amount of water in the tank, rounded to the nearest gallon, is 5127 gallons.

$$5: \begin{cases} 1: \text{identifies } t = 3 \text{ as a candidate} \\ 1: \text{integrand} \\ 1: \text{amount of water at } t = 3 \\ 1: \text{amount of water at } t = 7 \\ 1: \text{conclusion} \end{cases}$$

Question 2

At an intersection in Thomasville, Oregon, cars turn left at the rate $L(t) = 60\sqrt{t}\sin^2\left(\frac{t}{3}\right)$ cars per hour over the time interval $0 \le t \le 18$ hours. The graph of $y = L(t)$ is shown above.



(a) To the nearest whole number, find the total number of cars turning left at the intersection over the time interval $0 \le t \le 18$ hours.

(b) Traffic engineers will consider turn restrictions when $L(t) \ge 150$ cars per hour. Find all values of t for which $L(t) \ge 150$ and compute the average value of L over this time interval. Indicate units of measure.

(c) Traffic engineers will install a signal if there is any two-hour time interval during which the product of the total number of cars turning left and the total number of oncoming cars traveling straight through the intersection is greater than 200,000. In every two-hour time interval, 500 oncoming cars travel straight through the intersection. Does this intersection require a traffic signal? Explain the reasoning that leads to your conclusion.

(a) $\int_0^{18} L(t)\,dt \approx 1658$ cars

$2 : \begin{cases} 1 : \text{setup} \\ 1 : \text{answer} \end{cases}$

(b) $L(t) = 150$ when $t = 12.42831,\ 16.12166$
Let $R = 12.42831$ and $S = 16.12166$
$L(t) \ge 150$ for t in the interval $[R, S]$

$$\frac{1}{S-R}\int_R^S L(t)\,dt = 199.426 \text{ cars per hour}$$

$3 : \begin{cases} 1 : t\text{-interval when } L(t) \ge 150 \\ 1 : \text{average value integral} \\ 1 : \text{answer with units} \end{cases}$

(c) For the product to exceed 200,000, the number of cars turning left in a two-hour interval must be greater than 400.

$$\int_{13}^{15} L(t)\,dt = 431.931 > 400$$

OR

The number of cars turning left will be greater than 400 on a two-hour interval if $L(t) \ge 200$ on that interval.
$L(t) \ge 200$ on any two-hour subinterval of $[13.25304, 15.32386]$.

Yes, a traffic signal is required.

$4 : \begin{cases} 1 : \text{considers 400 cars} \\ 1 : \text{valid interval } [h, h+2] \\ 1 : \text{value of } \int_h^{h+2} L(t)\,dt \\ 1 : \text{answer and explanation} \end{cases}$

OR

$4 : \begin{cases} 1 : \text{considers 200 cars per hour} \\ 1 : \text{solves } L(t) \ge 200 \\ 1 : \text{discusses 2 hour interval} \\ 1 : \text{answer and explanation} \end{cases}$

Question 1

Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

(a) To the nearest whole number, how many cars pass through the intersection over the 30-minute period?

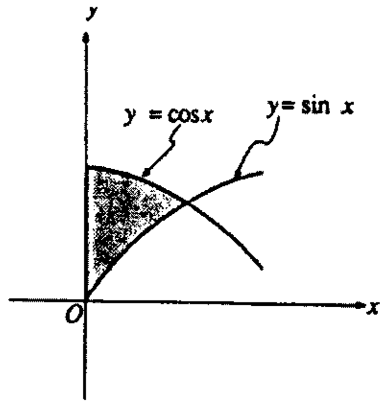
(b) Is the traffic flow increasing or decreasing at $t = 7$? Give a reason for your answer.

(c) What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

(d) What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

(a) $\int_0^{30} F(t)\,dt = 2474$ cars	$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) $F'(7) = -1.872$ or -1.873 Since $F'(7) < 0$, the traffic flow is decreasing at $t = 7$.	1 : answer with reason
(c) $\frac{1}{5}\int_{10}^{15} F(t)\,dt = 81.899$ cars/min	$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(d) $\frac{F(15) - F(10)}{15 - 10} = 1.517$ or 1.518 cars/min^2	1 : answer
Units of cars/min in (c) and cars/min^2 in (d)	1 : units in (c) and (d)

1991 BC3



Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of $y = \sin x$ and $y = \cos x$, as shown in the figure above.

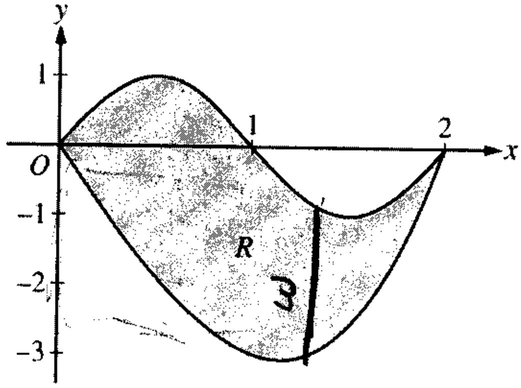
(a) Find the area of R .

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Find the volume of the solid whose base is R and whose cross sections cut by planes perpendicular to the x-axis are squares.

AP® CALCULUS BC
2008 SCORING GUIDELINES

Question 1



Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$ Area $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$	$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$ The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$	$2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \end{cases}$
(c) Volume $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 9.978$	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(d) Volume $= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369$ or 8.370	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

1992 AB5/BC2

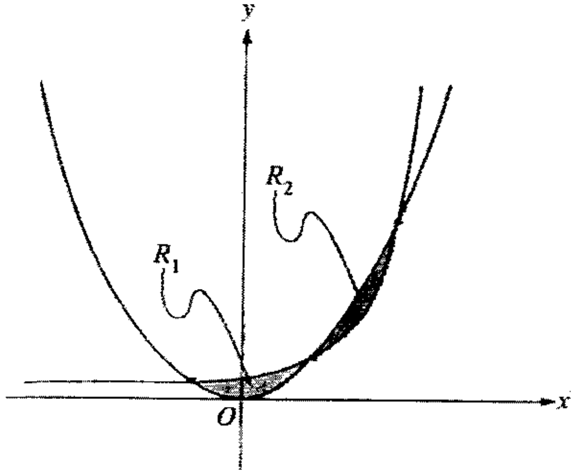
Let f be the function given by $f(x) = e^{-x}$, and let g be the function given by $g(x) = kx$, where k is the nonzero constant such that the graph of f is tangent to the graph of g.

(a) Find the x-coordinate of the point of tangency and the value of k.

(b) Let R be the region enclosed by the y-axis and the graphs of f and g. Using the results found in part (a), determine the area of R.

(c) Set up, but <u>do not integrate</u>, an integral expression in terms of a single variable for the volume of the solid generated by revolving the region R, given in part (b), about the <u>x-axis</u>.

1995 AB4/BC2



Note: Figure not drawn to scale.

The shaded regions R_1 and R_2 shown above are enclosed by the graphs of $f(x) = x^2$ and $g(x) = 2^x$.

(a) Find the x- and y-coordinates of the three points of intersection of the graphs of f and g.

(b) Without using absolute value, set up an expression involving one or more integrals that gives the total area enclosed by the graphs of f and g. Do not evaluate.

(c) Without using absolute value, set up an expression involving one or more integrals that gives the volume of the solid generated by revolving the region R_1 about the line $y = 5$. Do not evaluate.

1991 AB2

Let R be the region between the graphs of $y = 1 + \sin(\pi x)$ and $y = x^2$ from $x = 0$ to $x = 1$.

(a) Find the area of R.

(b) Set up, but do not integrate an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the x-axis.

(c) Set up, but do not integrate an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the y-axis.

1991 BC3
Solution

(a) $\text{Area} = \int_0^{\pi/4} \cos x - \sin x \, dx$

$$= (\sin x + \cos x)\Big|_0^{\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)$$

$$= \sqrt{2} - 1$$

(b) $V = \pi \int_0^{\pi/4} \cos^2 x - \sin^2 x \, dx$

$$= \pi \int_0^{\pi/4} \cos 2x \, dx$$

$$= \frac{\pi}{2} \sin 2x \Big|_0^{\pi/4}$$

$$= \frac{\pi}{2}(1-0) = \frac{\pi}{2}$$

(c) $V = \int_0^{\pi/4} (\cos x - \sin x)^2 \, dx$

$$= \int_0^{\pi/4} 1 - 2\sin x \cos x \, dx$$

$$= \left(x - \sin^2 x\right)\Big|_0^{\pi/4}$$

$$= \frac{\pi}{4} - \frac{1}{2} - (0-0)$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

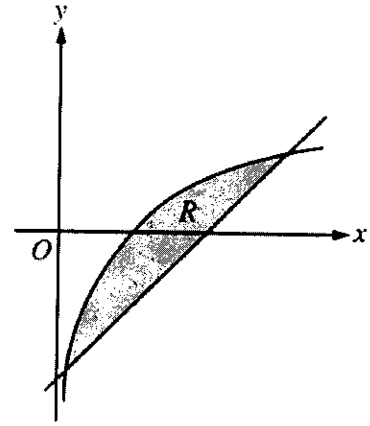
Question 1

Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$.

(c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the y-axis.



$\ln(x) = x - 2$ when $x = 0.15859$ and 3.14619.
Let $S = 0.15859$ and $T = 3.14619$

(a) Area of $R = \int_S^T (\ln(x) - (x-2))\,dx = 1.949$

$$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$$

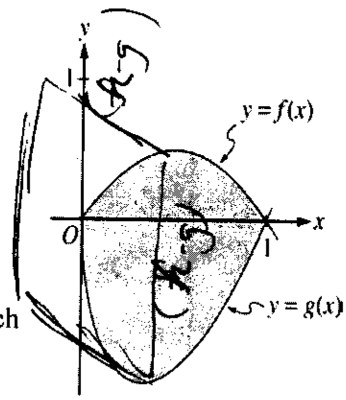
(b) Volume $= \pi \int_S^T \left((\ln(x)+3)^2 - (x-2+3)^2\right) dx$
$= 34.198$ or 34.199

$$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits, constant, and answer} \end{cases}$$

(c) Volume $= \pi \int_{S-2}^{T-2} \left((y+2)^2 - \left(e^y\right)^2\right) dy$

$$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$$

Question 2



Let f and g be the functions given by $f(x) = 2x(1-x)$ and $g(x) = 3(x-1)\sqrt{x}$ for $0 \le x \le 1$. The graphs of f and g are shown in the figure above.

(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Find the volume of the solid generated when the shaded region enclosed by the graphs of f and g is revolved about the horizontal line $y = 2$.

(c) Let h be the function given by $h(x) = kx(1-x)$ for $0 \le x \le 1$. For each $k > 0$, the region (not shown) enclosed by the graphs of h and g is the base of a solid with square cross sections perpendicular to the x-axis. There is a value of k for which the volume of this solid is equal to 15. Write, but do not solve, an equation involving an integral expression that could be used to find the value of k.

(a) Area $= \int_0^1 (f(x) - g(x))\,dx$

$= \int_0^1 \left(2x(1-x) - 3(x-1)\sqrt{x}\right) dx = 1.133$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_0^1 \left((2-g(x))^2 - (2-f(x))^2\right) dx$

$= \pi \int_0^1 \left(\left(2 - 3(x-1)\sqrt{x}\right)^2 - \left(2 - 2x(1-x)\right)^2\right) dx$

$= 16.179$

$4 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \\ \quad \langle -1 \rangle \text{ each error} \\ \quad \text{Note: } 0/2 \text{ if integral not of form} \\ \qquad c\int_a^b \left(R^2(x) - r^2(x)\right) dx \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \int_0^1 (h(x) - g(x))^2\,dx$

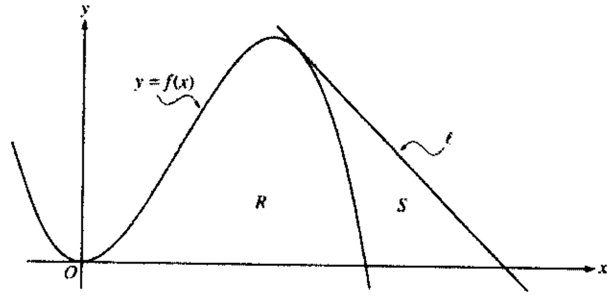
$\int_0^1 \left(kx(1-x) - 3(x-1)\sqrt{x}\right)^2 dx = 15$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

AP® CALCULUS BC
2003 SCORING GUIDELINES (Form B)

Question 1

Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.



(a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of S.

(c) Find the volume of the solid generated when R is revolved about the x-axis.

(a) $f'(x) = 8x - 3x^2$; $f'(3) = 24 - 27 = -3$

$f(3) = 36 - 27 = 9$

Tangent line at $x = 3$ is

$y = -3(x-3) + 9 = -3x + 18$,

which is the equation of line ℓ.

$$2 : \begin{cases} 1 : \text{finds } f'(3) \text{ and } f(3) \\ 1 : \begin{cases} \text{finds equation of tangent line} \\ \text{or} \\ \text{shows } (3,9) \text{ is on both the} \\ \text{graph of } f \text{ and line } \ell \end{cases} \end{cases}$$

(b) $f(x) = 0$ at $x = 4$

The line intersects the x-axis at $x = 6$.

$$\text{Area} = \frac{1}{2}(3)(9) - \int_3^4 \left(4x^2 - x^3\right) dx$$

$$= 7.916 \text{ or } 7.917$$

OR

$$\text{Area} = \int_3^4 \left((18 - 3x) - \left(4x^2 - x^3\right)\right) dx + \frac{1}{2}(2)(18 - 12)$$

$$= 7.916 \text{ or } 7.917$$

$$4 : \begin{cases} 2 : \text{integral for non-triangular region} \\ \quad 1 : \text{limits} \\ \quad 1 : \text{integrand} \\ 1 : \text{area of triangular region} \\ 1 : \text{answer} \end{cases}$$

(c) $$\text{Volume} = \pi \int_0^4 \left(4x^2 - x^3\right)^2 dx$$

$$= 156.038\pi \text{ or } 490.208$$

$$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

AP® CALCULUS BC
2007 SCORING GUIDELINES

Question 1

Let R be the region in the first and second quadrants bounded above by the graph of $y = \dfrac{20}{1+x^2}$ and below by the horizontal line $y = 2$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the x-axis.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles. Find the volume of this solid.

$\dfrac{20}{1+x^2} = 2$ when $x = \pm 3$	1 : correct limits in an integral in (a), (b), or (c)
(a) Area $= \displaystyle\int_{-3}^{3}\left(\frac{20}{1+x^2} - 2\right)dx = 37.961$ or 37.962	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) Volume $= \pi\displaystyle\int_{-3}^{3}\left(\left(\frac{20}{1+x^2}\right)^2 - 2^2\right)dx = 1871.190$	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(c) Volume $= \dfrac{\pi}{2}\displaystyle\int_{-3}^{3}\left(\frac{1}{2}\left(\frac{20}{1+x^2} - 2\right)\right)^2 dx$ $= \dfrac{\pi}{8}\displaystyle\int_{-3}^{3}\left(\frac{20}{1+x^2} - 2\right)^2 dx = 174.268$	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

1998 Calculus BC Scoring Guidelines

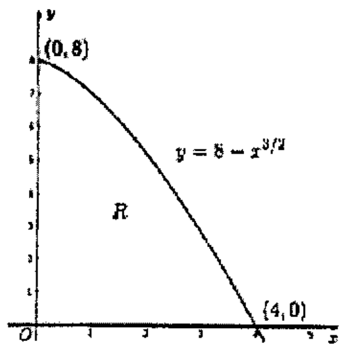
1. Let R be the region in the first quadrant bounded by the graph of $y = 8 - x^{\frac{3}{2}}$, the x-axis, and the y-axis.

   (a) Find the area of the region R.

   (b) Find the volume of the solid generated when R is revolved about the x-axis.

   (c) The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x-axis, they generate solids with equal volumes. Find the value of k.

(a)



$$A = \int_0^4 \left(8 - x^{3/2}\right) dx$$

$$= 8x - \frac{2}{5}x^{5/2}\Big|_0^4 = 32 - \frac{64}{5} = \frac{96}{5} = 19.2$$

$3 \begin{cases} 2\text{: integral} \\ \quad 1\text{: integrand} \\ \quad 1\text{: limits} \\ 1\text{: answer} \end{cases}$

(b) $V = \pi \int_0^4 \left(8 - x^{3/2}\right)^2 dx$

$$= \frac{576\pi}{5} = 115.2\pi \approx 361.911$$

$3 \begin{cases} 2\text{: integral} \\ \quad 1\text{: integrand} \\ \quad 1\text{: limits and constant} \\ 1\text{: answer} \end{cases}$

(c) $\pi \int_0^k \left(8 - x^{3/2}\right)^2 dx = \dfrac{115.2\pi}{2}$

$$\left[\begin{array}{l}\text{or} \\ \pi \int_0^k \left(8 - x^{3/2}\right)^2 dx = \pi \int_k^4 \left(8 - x^{3/2}\right)^2 dx\end{array}\right]$$

$$\int_0^k \left(8 - x^{3/2}\right)^2 dx = 57.6$$

$$\int_0^k \left(64 - 16x^{3/2} + x^3\right) dx = 57.6$$

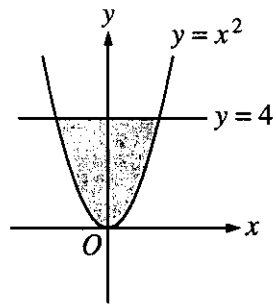
$$64k - \frac{32}{5}k^{5/2} + \frac{k^4}{4} = 57.6$$

$k \approx 0.995$ or 0.994

$3 \begin{cases} 1\text{: integral with } k \text{ in limits} \\ 1\text{: equates volumes} \\ 1\text{: answer} \end{cases}$

Note: 0/1 for answer in each part if no setup points earned

2. The shaded region, R, is bounded by the graph of $y = x^2$ and the line $y = 4$, as shown in the figure above.
   (a) Find the area of R.
   (b) Find the volume of the solid generated by revolving R about the x-axis.
   (c) There exists a number k, $k > 4$, such that when R is revolved about the line $y = k$, the resulting solid has the same volume as the solid in part (b). Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.



(a) $$\text{Area} = \int_{-2}^{2} (4 - x^2)\,dx = 2\int_0^2 (4 - x^2)\,dx = 2\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{32}{3} = 10.666 \text{ or } 10.667$$

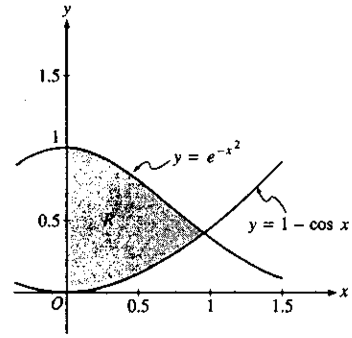
$2 \begin{cases} 1\text{: integral} \\ 1\text{: answer} \end{cases}$

(b) $$\text{Volume} = \pi\int_{-2}^{2} \left(4^2 - (x^2)^2\right)dx = 2\pi\int_0^2 (16 - x^4)\,dx = 2\pi\left[16x - \frac{x^5}{5}\right]_0^2 = \frac{256\pi}{5} = 160.849 \text{ or } 160.850$$

$3 \begin{cases} 1\text{: limits and constant} \\ 1\text{: integrand} \\ 1\text{: answer} \end{cases}$

(c) $$\pi\int_{-2}^{2} \left[(k - x^2)^2 - (k - 4)^2\right]dx = \frac{256\pi}{5}$$

$4 \begin{cases} 1\text{: limits and constant} \\ 2\text{: integrand} \\ \quad <-1> \text{ each error} \\ 1\text{: equation} \end{cases}$



Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the y-axis, as shown in the figure above.

(a) Find the area of the region R.

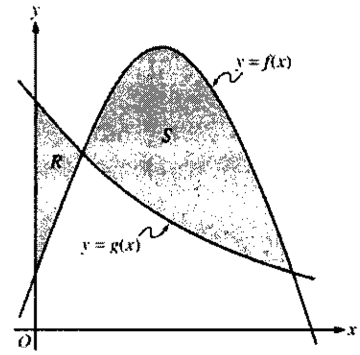
(b) Find the volume of the solid generated when the region R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

Region R $e^{-x^2} = 1 - \cos x$ at $x = 0.941944 = A$	1 : Correct limits in an integral in (a), (b), or (c).
(a) Area $= \int_0^A (e^{-x^2} - (1 - \cos x))\,dx$ $= 0.590$ or 0.591	2 { 1 : integrand 1 : answer
(b) Volume $= \pi \int_0^A \left(\left(e^{-x^2}\right)^2 - (1 - \cos x)^2 \right) dx$ $= 0.55596\pi = 1.746$ or 1.747	3 { 2 : integrand and constant < −1> each error 1 : answer
(c) Volume $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right)^2 dx$ $= 0.461$	3 { 2 : integrand < −1> each error Note: 0/2 if not of the form $k\int_c^d (f(x) - g(x))^2\,dx$ 1 : answer

Question 1

Let f and g be the functions given by $f(x) = \frac{1}{4} + \sin(\pi x)$ and $g(x) = 4^{-x}$. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of f and g, and let S be the shaded region in the first quadrant enclosed by the graphs of f and g, as shown in the figure above.



(a) Find the area of R.

(b) Find the area of S.

(c) Find the volume of the solid generated when S is revolved about the horizontal line $y = -1$.

$f(x) = g(x)$ when $\frac{1}{4} + \sin(\pi x) = 4^{-x}$.

f and g intersect when $x = 0.178218$ and when $x = 1$.
Let $a = 0.178218$.

(a) $\int_0^a (g(x) - f(x))\, dx = 0.064$ or 0.065

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $\int_a^1 (f(x) - g(x))\, dx = 0.410$

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $\pi \int_a^1 \left((f(x)+1)^2 - (g(x)+1)^2\right) dx = 4.558$ or 4.559

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits, constant, and answer} \end{cases}$

1993 AB3/BC1

Consider the curve $y^2 = 4 + x$ and chord AB joining the points $A(-4,0)$ and $B(0,2)$ on the curve.

(a) Find the x- and y-coordinates of the point on the curve where the tangent line is parallel to chord AB.

(b) Find the area of the region R enclosed by the curve and the chord AB.

(c) Find the volume of the solid generated when the region R, defined in part (b), is revolved about the x-axis.

1994 AB 2-BC 1

Let $\boldsymbol{R}$ be the region enclosed by the graphs of $y = e^x$, $y = x$, and the lines $\boldsymbol{x} = 0$ and $\boldsymbol{x} = 4$.

(a) Find the area of $\boldsymbol{R}$.

(b) Find the volume of the solid generated when $\boldsymbol{R}$ is revolved about the <u>$\boldsymbol{x}$-axis</u>.

(c) Set up, but <u>do not integrate</u>, an integral expression in terms of a single variable for the volume of the solid generated when $\boldsymbol{R}$ is revolved about the <u>$\boldsymbol{y}$-axis</u>.

1996 AB2

Let R be the region in the first quadrant under the graph of $y = \frac{1}{\sqrt{x}}$ for $4 \leq x \leq 9$.

(a) Find the area of R .

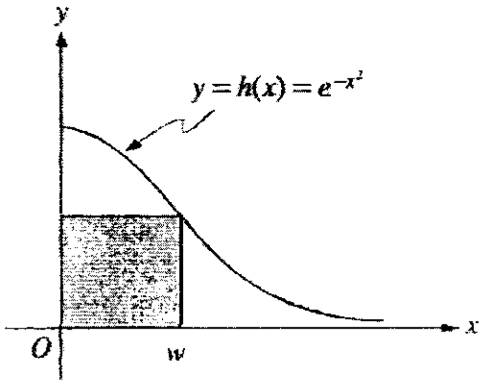
(b) If the line $x = k$ divides the region R into two regions of equal area, what is the value of k ?

(c) Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are squares.

1996 BC1

Consider the graph of the function h given by $h(x) = e^{-x^2}$ for $0 \le x < \infty$.

(a) Let R be the unbounded region in the first quadrant below the graph of h. Find the volume of the solid generated when R is revolved about the y-axis.

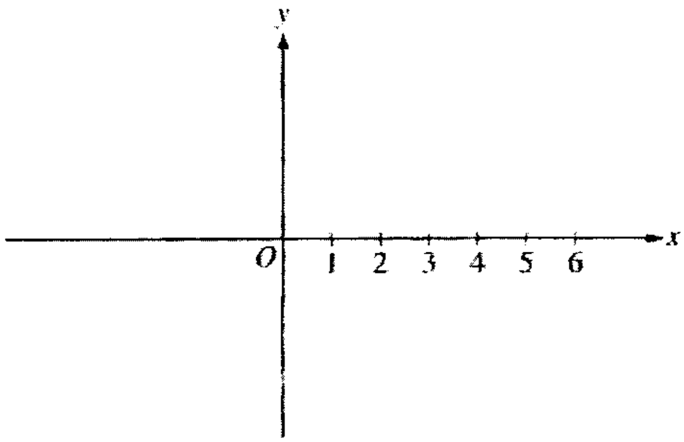


(b) Let $A(w)$ be the area of the shaded rectangle shown in the figure above. Show that $A(w)$ has its maximum value when w is the x-coordinate of the point of inflection of the graph of h.

1997 AB3

Let f be the function given by $f(x) = \sqrt{x-3}$.

(a) On the axes provided below, sketch the graph of f and shade the region R enclosed by the graph of f, the x-axis, and the vertical line $x = 6$.



(b) Find the area of the region R described in part (a).

(c) Rather than using the line $x = 6$ as in part (a), consider the line $x = w$, where w can be any number greater than 3. Let $A(w)$ be the area of the region enclosed by the graph of f, the x-axis, and the vertical line $x = w$. Write an integral expression for $A(w)$.

(d) Let $A(w)$ be as described in part (c). Find the rate of change of A with respect to w when $w = 6$.

1997 BC3

Let R be the region enclosed by the graphs of $y = \ln\left(x^2 + 1\right)$ and $y = \cos x$.

(a) Find the area of R.

(b) Write an expression involving one or more integrals that gives the length of the boundary of the region R. Do not evaluate.

(c) The base of a solid is the region R. Each cross section of the solid perpendicular to the x-axis is an equilateral triangle. Write an expression involving one or more integrals that gives the volume of the solid. Do not evaluate.

1989 AB2

Let R be the region in the first quadrant enclosed by the graph of $y = \sqrt{6x+4}$, the line $y = 2x$, and the y-axis.

(a) Find the area of R.

(b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the x-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the y-axis.

1989 BC2

Let R be the region enclosed by the graph of $y = \frac{x^2}{x^2+1}$, the line $x = 1$, and the x-axis.

(a) Find the area of R .

(b) Find the volume of the solid generated when R is rotated about the y-axis.

1990 AB3

Let R be the region enclosed by the graphs of $y = e^x$, $y = (x-1)^2$, and the line $x = 1$.

(a) Find the area of R .

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the y-axis.

1990 BC2

Let R be the region in the xy-plane between the graphs of $y = e^x$ and $y = e^{-x}$ from $x = 0$ to $x = 2$.

(a) Find the volume of the solid generated when R is revolved about the x-axis.

(b) Find the volume of the solid generated when R is revolved about the y-axis.

1989 AB6

Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well; that is, $\frac{dy}{dt} = ky$, where y is the amount of oil left in the well at any time t. Initially there were 1,000,000 gallons of oil in the well, and 6 years later there were 500,000 gallons remaining. It will no longer be profitable to pump oil when there are fewer than 50,000 gallons remaining.

(a) Write an equation for y, the amount of oil remaining in the well at any time t.

(b) At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining?

(c) In order not to lose money, at what time t should oil no longer be pumped from the well?

1996 AB3/BC3

The rate of consumption of cola in the United States is given by $S(t) = Ce^{kt}$, where S is measured in billions of gallons per year and t is measured in years from the beginning of 1980.

(a) The consumption rate doubles every 5 years and the consumption rate at the beginning of 1980 was 6 billion gallons per year. Find C and k.

(b) Find the average rate of consumption of cola over the 10-year time period beginning January 1, 1983. Indicate units of measure.

(c) Use the trapezoidal rule with four equal subdivisions to estimate $\int_5^7 S(t)\,dt$.

(d) Using correct units, explain the meaning of $\int_5^7 S(t)\,dt$ in terms of cola consumption.

1993 BC6

Let f be a function that is differentiable throughout its domain and that has the following properties.

(i) $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)\,f(y)}$ for all real numbers x, y, and $x+y$ in the domain of f

(ii) $\lim\limits_{h\to 0} f(h)=0$

(iii) $\lim\limits_{h\to 0}\dfrac{f(h)}{h}=1$

(a) Show that $f(0)=0$.

(b) Use the definition of the derivative to show that $f'(x)=1+[f(x)]^2$. Indicate clearly where properties (i), (ii), and (iii) are used.

(c) Find $f(x)$ by solving the differential equation in part (b).

1993 AB6

Let $P(t)$ represent the number of wolves in a population at time t years, when $t \geq 0$. The population $P(t)$ is increasing at a rate directly proportional to $800 - P(t)$, where the constant of proportionality is k.

(a) If $P(0) = 500$, find $P(t)$ in terms of t and k.

(b) If $P(2) = 700$, find k.

(c) Find $\lim_{t \to \infty} P(t)$.

1989 BC5

At any time $t \geq 0$, the velocity of a particle traveling along the x-axis is given by the differential equation $\frac{dx}{dt} - 10x = 60e^{4t}$.

(a) Find the general solution $x(t)$ for the position of the particle.

(b) If the position of the particle at time $t = 0$ is $x = -8$, find the particular solution $x(t)$ for the position of the particle.

(c) Use the particular solution from part (b) to find the time at which the particle is at rest.

1991 BC6

A certain rumor spreads through a community at the rate $\frac{dy}{dt} = 2y(1-y)$, where y is the proportion of the population that has heard the rumor at time t.

(a) What proportion of the population has heard the rumor when it is spreading the fastest?

(b) If at time $t = 0$ ten percent of the people have heard the rumor, find y as a function of t.

(c) At what time t is the rumor spreading the fastest?

Question 5

Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$ and $\lim_{x\to\infty} f(x) = 0$.

(a) Evaluate $\int_1^\infty -3xf(x)\,dx$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 1$ with a step size of 0.5, to approximate $f(2)$.

(c) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = -3xy$ with the initial condition $f(1) = 4$.

(a) $\int_1^\infty -3xf(x)\,dx$

$= \int_1^\infty f'(x)\,dx = \lim_{b\to\infty} \int_1^b f'(x)\,dx = \lim_{b\to\infty} f(x)\Big|_1^b$

$= \lim_{b\to\infty} f(b) - f(1) = 0 - 4 = -4$

$2: \begin{cases} 1: \text{use of FTC} \\ 1: \text{answer from limiting process} \end{cases}$

(b) $f(1.5) \approx f(1) + f'(1)(0.5)$

$= 4 - 3(1)(4)(0.5) = -2$

$f(2) \approx -2 + f'(1.5)(0.5)$

$\approx -2 - 3(1.5)(-2)(0.5) = 2.5$

$2: \begin{cases} 1: \text{Euler's method equations or equivalent table} \\ 1: \text{Euler approximation to } f(2) \\ \quad \text{(not eligible without first point)} \end{cases}$

(c) $\frac{1}{y}\,dy = -3x\,dx$

$\ln y = -\frac{3}{2}x^2 + k$

$y = Ce^{-\frac{3}{2}x^2}$

$4 = Ce^{-\frac{3}{2}}$; $C = 4e^{\frac{3}{2}}$

$y = 4e^{\frac{3}{2}}e^{-\frac{3}{2}x^2}$

$5: \begin{cases} 1: \text{separates variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition } f(1) = 4 \\ 1: \text{solves for } y \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

Question 5

A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t\to\infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t\to\infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function Y found in part (c), what is $\lim_{t\to\infty} Y(t)$?

Solution	Scoring
(a) For this logistic differential equation, the carrying capacity is 12. If $P(0) = 3$, $\lim_{t\to\infty} P(t) = 12$. If $P(0) = 20$, $\lim_{t\to\infty} P(t) = 12$.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{answer} \end{cases}$
(b) The population is growing the fastest when P is half the carrying capacity. Therefore, P is growing the fastest when $P = 6$.	1 : answer
(c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$ $\ln\lvert Y\rvert = \frac{t}{5} - \frac{t^2}{120} + C$ $Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$ $K = 3$ $Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$	$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } Y \\ \quad 0/1 \text{ if } Y \text{ is not exponential} \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables
(d) $\lim_{t\to\infty} Y(t) = 0$	1 : answer $0/1$ if Y is not exponential

AP® CALCULUS BC
2002 SCORING GUIDELINES (Form B)

Question 5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{3-x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. Find the x-coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

(b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

(a) $\dfrac{dy}{dx} = 0$ when $x = 3$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,-2)} = \left.\frac{-y - y'(3-x)}{y^2}\right|_{(3,-2)} = \frac{1}{2},$$

so f has a local minimum at this point.

or

Because f is continuous for $1 < x < 5$, there is an interval containing $x = 3$ on which $y < 0$. On this interval, $\dfrac{dy}{dx}$ is negative to the left of $x = 3$ and $\dfrac{dy}{dx}$ is positive to the right of $x = 3$. Therefore f has a local minimum at $x = 3$.

$$3\begin{cases} 1 : x = 3 \\ 1 : \text{local minimum} \\ 1 : \text{justification} \end{cases}$$

(b) $y\,dy = (3 - x)\,dx$

$$\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + C$$

$$8 = 18 - 18 + C\ ;\ C = 8$$

$$y^2 = 6x - x^2 + 16$$

$$y = -\sqrt{6x - x^2 + 16}$$

$$6\begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivative of } dy \text{ term} \\ 1 : \text{antiderivative of } dx \text{ term} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } g(6) = -4 \\ 1 : \text{solves for } y \end{cases}$$

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

(a) $y^2 + 2xy\dfrac{dy}{dx} - 3x^2y - x^3\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}\left(2xy - x^3\right) = 3x^2y - y^2$

$\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$

$2\begin{cases} 1: \text{implicit differentiation} \\ 1: \text{verifies expression for } \dfrac{dy}{dx} \end{cases}$

(b) When $x = 1$, $y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3,\ y = -2$

At $(1,3)$, $\dfrac{dy}{dx} = \dfrac{9-9}{6-1} = 0$

Tangent line equation is $y = 3$

At $(1,-2)$, $\dfrac{dy}{dx} = \dfrac{-6-4}{-4-1} = \dfrac{-10}{-5} = 2$

Tangent line equation is $y + 2 = 2(x-1)$

$4\begin{cases} 1: \ y^2 - y = 6 \\ 1: \ \text{solves for } y \\ 2: \ \text{tangent lines} \end{cases}$

Note: 0/4 if not solving an equation of the form $y^2 - y = k$

(c) Tangent line is vertical when $2xy - x^3 = 0$

$x\left(2y - x^2\right) = 0$ gives $x = 0$ or $y = \dfrac{1}{2}x^2$

There is no point on the curve with x-coordinate 0.

When $y = \dfrac{1}{2}x^2$, $\dfrac{1}{4}x^5 - \dfrac{1}{2}x^5 = 6$

$-\dfrac{1}{4}x^5 = 6$

$x = \sqrt[5]{-24}$

$3\begin{cases} 1: \ \text{sets denominator of } \dfrac{dy}{dx} \text{ equal to } 0 \\ 1: \ \text{substitutes } y = \dfrac{1}{2}x^2 \text{ or } x = \pm\sqrt{2y} \\ \quad\ \text{into the equation for the curve} \\ 1: \ \text{solves for } x\text{-coordinate} \end{cases}$

Question 5

Consider the differential equation $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$ for $y \neq 2$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = -4$.

(a) Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(-1, -4)$.

(b) Is it possible for the x-axis to be tangent to the graph of f at some point? Explain why or why not.

(c) Find the second-degree Taylor polynomial for f about $x = -1$.

(d) Use Euler's method, starting at $x = -1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(a) $\left.\frac{dy}{dx}\right|_{(-1,-4)} = 6$

$\frac{d^2y}{dx^2} = 10x + 6(y-2)^{-2}\frac{dy}{dx}$

$\left.\frac{d^2y}{dx^2}\right|_{(-1,-4)} = -10 + 6\frac{1}{(-6)^2}6 = -9$

$3 : \begin{cases} 1 : \left.\frac{dy}{dx}\right|_{(-1,-4)} \\ 1 : \frac{d^2y}{dx^2} \\ 1 : \left.\frac{d^2y}{dx^2}\right|_{(-1,-4)} \end{cases}$

(b) The x-axis will be tangent to the graph of f if $\left.\frac{dy}{dx}\right|_{(k,0)} = 0$.

The x-axis will never be tangent to the graph of f because $\left.\frac{dy}{dx}\right|_{(k,0)} = 5k^2 + 3 > 0$ for all k.

$2 : \begin{cases} 1 : \frac{dy}{dx} = 0 \text{ and } y = 0 \\ 1 : \text{answer and explanation} \end{cases}$

(c) $P(x) = -4 + 6(x+1) - \frac{9}{2}(x+1)^2$

$2 : \begin{cases} 1 : \text{quadratic and centered at } x = -1 \\ 1 : \text{coefficients} \end{cases}$

(d) $f(-1) = -4$

$f\left(-\frac{1}{2}\right) \approx -4 + \frac{1}{2}(6) = -1$

$f(0) \approx -1 + \frac{1}{2}\left(\frac{5}{4} + 2\right) = \frac{5}{8}$

$2 : \begin{cases} 1 : \text{Euler's method with 2 steps} \\ 1 : \text{Euler's approximation to } f(0) \end{cases}$

Question 4

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

(a) $$2x + 8yy' = 3y + 3xy'$$
$$(8y - 3x)y' = 3y - 2x$$
$$y' = \frac{3y - 2x}{8y - 3x}$$

$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$

(b) $\frac{3y - 2x}{8y - 3x} = 0;\ 3y - 2x = 0$

When $x = 3$, $3y = 6$
$y = 2$

$3^2 + 4 \cdot 2^2 = 25$ and $7 + 3 \cdot 3 \cdot 2 = 25$

Therefore, $P = (3, 2)$ is on the curve and the slope is 0 at this point.

$3 : \begin{cases} 1 : \frac{dy}{dx} = 0 \\ 1 : \text{shows slope is 0 at } (3, 2) \\ 1 : \text{shows } (3, 2) \text{ lies on curve} \end{cases}$

(c) $$\frac{d^2y}{dx^2} = \frac{(8y - 3x)(3y' - 2) - (3y - 2x)(8y' - 3)}{(8y - 3x)^2}$$

At $P = (3, 2)$, $\frac{d^2y}{dx^2} = \frac{(16 - 9)(-2)}{(16 - 9)^2} = -\frac{2}{7}$.

Since $y' = 0$ and $y'' < 0$ at P, the curve has a local maximum at P.

$4 : \begin{cases} 2 : \frac{d^2y}{dx^2} \\ 1 : \text{value of } \frac{d^2y}{dx^2} \text{ at } (3, 2) \\ 1 : \text{conclusion with justification} \end{cases}$

1994 BC 5

Let f be the function given by $f(x) = e^{-2x^2}$.

(a) Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x = 0$.

(b) Find the interval of convergence of the power series for $f(x)$ about $x = 0$. Show the analysis that leads to your conclusion.

(c) Let g be the function given by the sum of the first four nonzero terms of the power series for $f(x)$ about $x = 0$. Show that $|f(x) - g(x)| < 0.02$ for $-0.6 \le x \le 0.6$.

1997 BC2

Let $P(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ be the fourth-degree Taylor polynomial for the function f about 4. Assume f has derivatives of all orders for all real numbers.

(a) Find $f(4)$ and $f'''(4)$.

(b) Write the second-degree Taylor polynomial for f' about 4 and use it to approximate $f'(4.3)$.

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)\,dt$ about 4.

(d) Can $f(3)$ be determined from the information given? Justify your answer.

1996 BC2

The Maclaurin series for $f(x)$ is given by $1+\dfrac{x}{2!}+\dfrac{x^2}{3!}+\dfrac{x^3}{4!}+\cdots+\dfrac{x^n}{(n+1)!}+\cdots$

(a) Find $f'(0)$ and $f^{(17)}(0)$.

(b) For what values of x does the given series converge? Show your reasoning.

(c) Let $g(x)=x\,f(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.

(d) Write $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same familiar function.

1995 BC4

Let f be a function that has derivatives of all orders for all real numbers. Assume $f(1)=3, f'(1)=-2, f''(1)=2$, and $f'''(1)=4$.

(a) Write the second-degree Taylor polynomial for f about $x=1$ and use it to approximate $f(0.7)$.

(b) Write the third-degree Taylor polynomial for f about $x=1$ and use it to approximate $f(1.2)$.

(c) Write the second-degree Taylor polynomial for f', the derivative of f, about $x=1$ and use it to approximate $f'(1.2)$.

1993 BC5

Let f be the function given by $f(x) = e^{\frac{x}{2}}$.

(a) Write the first four nonzero terms and the general term for the Taylor series expansion of $f(x)$ about $x = 0$.

(b) Use the result from part (a) to write the first three nonzero terms and the general term of the series expansion about $x = 0$ for $g(x) = \dfrac{e^{\frac{x}{2}} - 1}{x}$.

(c) For the function g in part (b), find $g'(2)$ and use it to show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$.

1992 BC6

Consider the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$, where $p \geq 0$.

(a) Show that the series converges for $p > 1$.

(b) Determine whether the series converges or diverges for $p = 1$. Show your analysis.

(c) Show that the series diverges for $0 \leq p < 1$.

1991 BC5

Let f be the function given by $f(t) = \frac{4}{1+t^2}$ and G be the function given by $G(x) = \int_0^x f(t)\, dt$.

(a) Find the first four nonzero terms and the general term for the power series expansion of $f(t)$ about $t = 0$.

(b) Find the first four nonzero terms and the general term for the power series expansion of $G(x)$ about $x = 0$.

(c) Find the interval of convergence of the power series in part (b). (Your solution must include an analysis that justifies your answer.)

1990 BC5

Let f be the function defined by $f(x) = \dfrac{1}{x-1}$.

(a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.

(b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.

(c) Use the series in part (b) to compute a number that differs from $\ln\frac{3}{2}$ by less than 0.05. Justify your answer.

Question 6

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Find the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(c) Give a value of p such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

(a) $\ln\left(\frac{1}{1+3x}\right) = \ln\left(\frac{1}{1-(-3x)}\right)$

$= \sum_{n=1}^{\infty} \frac{(-3x)^n}{n}$ or $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} x^n$

We must have $-1 \le -3x < 1$, so interval of convergence is $-\frac{1}{3} < x \le \frac{1}{3}$.

$2 \begin{cases} 1 : \text{series} \\ 1 : \text{interval of convergence} \end{cases}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\frac{1}{1-(-1)}\right) = \ln\left(\frac{1}{2}\right)$

1 : answer

(c) Some p such that $0 < p \le \frac{1}{2}$ because $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges by AST, but the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges for $2p \le 1$.

$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{(-1)^n}{n^p} \text{ converges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ diverges} \end{cases}$

(d) Some p such that $\frac{1}{2} < p \le 1$ because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \le 1$ and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges for $2p > 1$.

$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{1}{n^p} \text{ diverges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ converges} \end{cases}$

The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about $x = 5$.

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

(a) $f'(5) = \dfrac{-1!}{2(3)}$, $f''(5) = \dfrac{2!}{4(4)}$, $f'''(5) = \dfrac{-3!}{8(5)}$

$$P_3(f,5)(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$$

3 : $P_3(f,5)(x)$
$\quad <-1>$ each error or missing term

Note: $<-1>$ max for improper use of extra terms, equality or $+\ldots$

(b) $a_n = \dfrac{f^{(n)}(5)}{n!} = \dfrac{(-1)^n}{2^n (n+2)}$

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)}}{\dfrac{(-1)^n(x-5)^n}{2^n(n+2)}}\right| = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+2}{n+3}\right)|x-5|$$

$$= \frac{|x-5|}{2} < 1$$

The radius of convergence is 2.

$$4\begin{cases} 1: \text{ general term} \\ 1: \text{ sets up ratio test} \\ 1: \text{ computes the limit} \\ 1: \text{ applies ratio test to get radius of convergence} \end{cases}$$

(c) The Taylor series about $x = 5$ for the function f, when evaluated at $x = 6$, is an alternating series with absolute value of terms decreasing to 0. The error in approximating $f(6)$ with the 6th degree Taylor polynomial at $x = 6$ is less than the first omitted term in the series.

$$|f(6) - P_6(f,5)(6)| \le \frac{1}{2^7(9)} = \frac{1}{1152} < \frac{1}{1000}$$

$$2\begin{cases} 1: \text{ error bound} < \dfrac{1}{1000} \\ 1: \text{ refers to an alternating series and indicates the error bound is found from the next term} \end{cases}$$

4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.
   (a) Write the third–degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.
   (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.
   (c) Write the fourth–degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

(a) $T_3(f,2)(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3$

$$f(1.5) \approx T_3(f,2)(1.5)$$
$$= -3 + 5(-0.5) + \frac{3}{2}(-0.5)^2 - \frac{4}{3}(-0.5)^3$$
$$= -4.958\overline{3} = -4.958$$

$4 \begin{cases} 3: T_3(f,2)(x) \\ \quad <-1> \text{ each error} \\ 1: \text{approximation of } f(1.5) \end{cases}$

(b) Lagrange Error Bound $= \frac{3}{4!}|1.5 - 2|^4 = 0.0078125$

$f(1.5) > -4.958\overline{3} - 0.0078125 = -4.966 > -5$

Therefore, $f(1.5) \neq -5$.

$2 \begin{cases} 1: \text{value of Lagrange Error Bound} \\ 1: \text{explanation} \end{cases}$

(c) $P(x) = T_4(g,0)(x)$

$$= T_2(f,2)(x^2+2) = -3 + 5x^2 + \frac{3}{2}x^4$$

The coefficient of x in $P(x)$ is $g'(0)$. This coefficient is 0, so $g'(0) = 0$.

The coefficient of x^2 in $P(x)$ is $\frac{g''(0)}{2!}$. This coefficient is 5, so $g''(0) = 10$ which is greater than 0.

Therefore, g has a relative minimum at $x = 0$.

$3 \begin{cases} 2: T_4(g,0)(x) \\ \quad <-1> \text{ each incorrect, missing, or extra term} \\ 1: \text{explanation} \end{cases}$

Note:
$<-1>$ max for improper use of $+\ldots$ or equality

1998 Calculus BC Scoring Guidelines

3. Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 5$, $f'(0) = -3$, $f''(0) = 1$, and $f'''(0) = 4$.
   (a) Write the third-degree Taylor polynomial for f about $x = 0$ and use it to approximate $f(0.2)$.
   (b) Write the fourth-degree Taylor polynomial for g, where $g(x) = f(x^2)$, about $x = 0$.
   (c) Write the third-degree Taylor polynomial for h, where $h(x) = \int_0^x f(t)\,dt$, about $x = 0$.
   (d) Let h be defined as in part (c). Given that $f(1) = 3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

(a) $P_3(f)(x) = 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3$ $f(0.2) \approx P_3(f)(0.2) =$ $5 - 3(0.2) + \frac{0.04}{2} + \frac{2(0.008)}{3} =$ 4.425	$3\begin{cases} 2\text{: } 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3 \\ \quad <-1> \text{ each incorrect term,} \\ \quad \text{extra term, or } + \cdots \\ 1\text{: approximates } f(0.2) \end{cases}$ $<-1>$ for incorrect use of $=$
(b) $P_4(g)(x) = P_2(f)(x^2) = 5 - 3x^2 + \frac{1}{2}x^4$	2: $P_2(f)(x^2)$ $<-1>$ each incorrect or extra term
(c) $P_3(h)(x) = \int_0^x \left(5 - 3t + \frac{1}{2}t^2\right) dt$ $= \left[5t - \frac{3}{2}t^2 + \frac{1}{6}t^3\right]_0^x$ $= 5x - \frac{3}{2}x^2 + \frac{1}{6}x^3$	$2\begin{cases} 1\text{: } P_3(h)(x) = \int_0^x P_2(f)(t)\,dt \\ 1\text{: answer} \\ \quad 0/1 \text{ if any incorrect or extra terms} \end{cases}$
(d) $h(1) = \int_0^1 f(t)\,dt$ cannot be determined because $f(t)$ is known only for $t = 0$ and $t = 1$	$2\begin{cases} 1\text{: } h(1) \text{ cannot be determined} \\ 1\text{: reason} \end{cases}$

AP® CALCULUS BC
2008 SCORING GUIDELINES

Question 3

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x-2)$, so $h(1.9) \approx P_1(1.9) = 67.2$

$P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.

$4 : \begin{cases} 2 : P_1(x) \\ 1 : P_1(1.9) \\ 1 : P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$

(b) $P_3(x) = 80 + 128(x-2) + \frac{488}{6}(x-2)^2 + \frac{448}{18}(x-2)^3$

$h(1.9) \approx P_3(1.9) = 67.988$

$3 : \begin{cases} 2 : P_3(x) \\ 1 : P_3(1.9) \end{cases}$

(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max\limits_{1.9 \le x \le 2} \left|h^{(4)}(x)\right| = \frac{584}{9}$.

Therefore, $|h(1.9) - P_3(1.9)| \le \frac{584}{9} \frac{|1.9-2|^4}{4!}$

$= 2.7037 \times 10^{-4}$

$< 3 \times 10^{-4}$

$2 : \begin{cases} 1 : \text{form of Lagrange error estimate} \\ 1 : \text{reasoning} \end{cases}$

Question 6

Let f be the function given by $f(x) = e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use your answer to part (a) to find $\lim\limits_{x\to 0} \dfrac{1 - x^2 - f(x)}{x^4}$.

(c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\,dt$.

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\,dt$ by less than $\dfrac{1}{200}$.

(a) $$e^{-x^2} = 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$$
$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$$

$3 : \begin{cases} 1 : \text{two of } 1, -x^2, \frac{x^4}{2}, -\frac{x^6}{6} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(b) $$\frac{1 - x^2 - f(x)}{x^4} = -\frac{1}{2} + \frac{x^2}{6} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1} x^{2n-4}}{n!}$$

Thus, $\lim\limits_{x\to 0}\left(\dfrac{1 - x^2 - f(x)}{x^4}\right) = -\dfrac{1}{2}$.

$1 : \text{answer}$

(c) $$\int_0^x e^{-t^2}\,dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots\right) dt$$
$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$$

Using the first two terms of this series, we estimate that $\int_0^{1/2} e^{-t^2}\,dt \approx \frac{1}{2} - \left(\frac{1}{3}\right)\left(\frac{1}{8}\right) = \frac{11}{24}$.

$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{estimate} \end{cases}$

(d) $\left|\int_0^{1/2} e^{-t^2}\,dt - \frac{11}{24}\right| < \left(\frac{1}{2}\right)^5 \cdot \frac{1}{10} = \frac{1}{320} < \frac{1}{200}$, since

$\int_0^{1/2} e^{-t^2}\,dt = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}$, which is an alternating series with individual terms that decrease in absolute value to 0.

$2 : \begin{cases} 1 : \text{uses the third term as the error bound} \\ 1 : \text{explanation} \end{cases}$

Question 6

The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

(a) $$\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n nx^n}\right| = \frac{(n+1)^2}{(n+2)(n)} \cdot |x|$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{(n+2)(n)} \cdot |x| = |x|$$

The series converges when $-1 < x < 1$.

When $x = 1$, the series is $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

When $x = -1$, the series is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

Thus, the interval of convergence is $-1 < x < 1$.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies radius of convergence
- 1 : considers both endpoints
- 1 : analysis/conclusion for both endpoints

(b) $f'(x) = -\frac{1}{2} + \frac{4}{3}x - \frac{9}{4}x^2 + \cdots$ and $f'(0) = -\frac{1}{2}$.

$g'(x) = -\frac{1}{2!} + \frac{2}{4!}x - \frac{3}{6!}x^2 + \cdots$ and $g'(0) = -\frac{1}{2}$.

$y'(0) = f'(0) - g'(0) = 0$

$f''(0) = \frac{4}{3}$ and $g''(0) = \frac{2}{4!} = \frac{1}{12}$.

Thus, $y''(0) = \frac{4}{3} - \frac{1}{12} > 0$.

Since $y'(0) = 0$ and $y''(0) > 0$, y has a relative minimum at $x = 0$.

4 :
- 1 : $y'(0)$
- 1 : $y''(0)$
- 1 : conclusion
- 1 : reasoning

Question 6

Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \frac{(n-1)!}{3^n}$.

(a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

(b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

(c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(a) $P_6(x) = 7 + \frac{1!}{3^2}\cdot\frac{1}{2!}(x-2)^2 + \frac{3!}{3^4}\cdot\frac{1}{4!}(x-2)^4 + \frac{5!}{3^6}\cdot\frac{1}{6!}(x-2)^6$

$3 : \begin{cases} 1 : \text{polynomial about } x = 2 \\ 2 : P_6(x) \\ \quad \langle -1 \rangle \text{ each incorrect term} \\ \quad \langle -1 \rangle \text{ max for all extra terms,} \\ \qquad + \cdots, \text{ misuse of equality} \end{cases}$

(b) $\frac{(2n-1)!}{3^{2n}}\cdot\frac{1}{(2n)!} = \frac{1}{3^{2n}(2n)}$

1 : coefficient

(c) The Taylor series for f about $x = 2$ is

$$f(x) = 7 + \sum_{n=1}^{\infty}\frac{1}{2n\cdot 3^{2n}}(x-2)^{2n}.$$

$$L = \lim_{n\to\infty}\left|\frac{\frac{1}{2(n+1)}\cdot\frac{1}{3^{2(n+1)}}(x-2)^{2(n+1)}}{\frac{1}{2n}\cdot\frac{1}{3^{2n}}(x-2)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n}{2(n+1)}\cdot\frac{3^{2n}}{3^2 3^{2n}}(x-2)^2\right| = \frac{(x-2)^2}{9}$$

$L < 1$ when $|x-2| < 3$.

Thus, the series converges when $-1 < x < 5$.

When $x = 5$, the series is $7 + \sum_{n=1}^{\infty}\frac{3^{2n}}{2n\cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, diverges.

When $x = -1$, the series is $7 + \sum_{n=1}^{\infty}\frac{(-3)^{2n}}{2n\cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, diverges.

The interval of convergence is $(-1, 5)$.

$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{identifies interior of} \\ \quad \text{interval of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis/conclusion for} \\ \quad \text{both endpoints} \end{cases}$

Question 6

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$.

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

(a) $f(0) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(0) = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$

$f''(0) = -25\sin\left(\frac{\pi}{4}\right) = -\frac{25\sqrt{2}}{2}$

$f'''(0) = -125\cos\left(\frac{\pi}{4}\right) = -\frac{125\sqrt{2}}{2}$

$P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$

4 : $P(x)$

$\langle -1 \rangle$ each error or missing term

deduct only once for $\sin\left(\frac{\pi}{4}\right)$ evaluation error

deduct only once for $\cos\left(\frac{\pi}{4}\right)$ evaluation error

$\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality

(b) $\frac{-5^{22}\sqrt{2}}{2(22!)}$

$2 : \begin{cases} 1 : \text{magnitude} \\ 1 : \text{sign} \end{cases}$

(c) $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| \le \max\limits_{0 \le c \le \frac{1}{10}} \left|f^{(4)}(c)\right| \left(\frac{1}{4!}\right)\left(\frac{1}{10}\right)^4$

$\le \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100}$

1 : error bound in an appropriate inequality

(d) The third-degree Taylor polynomial for G about $x = 0$ is $\int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2\right) dt$

$= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$

2 : third-degree Taylor polynomial for G about $x = 0$

$\langle -1 \rangle$ each incorrect or missing term

$\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality

Question 6

The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n+1)!}{3^n}$ for $n \geq 1$, and $f(2) = 1$.

(a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

(b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x = 2$.

(d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

(a) $f(2) = 1$; $f'(2) = \dfrac{2!}{3}$; $f''(2) = \dfrac{3!}{3^2}$; $f'''(2) = \dfrac{4!}{3^3}$

$$f(x) = 1 + \frac{2}{3}(x-2) + \frac{3!}{2!3^2}(x-2)^2 + \frac{4!}{3!3^3}(x-2)^3 + \cdots + \frac{(n+1)!}{n!3^n}(x-2)^n + \cdots$$
$$= 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \cdots + \frac{n+1}{3^n}(x-2)^n + \cdots$$

3 :
- 1 : coefficients $\dfrac{f^{(n)}(2)}{n!}$ in first four terms
- 1 : powers of $(x-2)$ in first four terms
- 1 : general term

(b) $$\lim_{n\to\infty}\left|\frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{1}{3}|x-2|$$
$$= \frac{1}{3}|x-2| < 1 \text{ when } |x-2| < 3$$

The radius of convergence is 3.

3 :
- 1 : sets up ratio
- 1 : limit
- 1 : applies ratio test to conclude radius of convergence is 3

(c) $g(2) = 3$; $g'(2) = f(2)$; $g''(2) = f'(2)$; $g'''(2) = f''(2)$

$$g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \cdots + \frac{1}{3^n}(x-2)^{n+1} + \cdots$$

2 :
- 1 : first four terms
- 1 : general term

(d) No, the Taylor series does not converge at $x = -2$ because the geometric series only converges on the interval $|x-2| < 3$.

1 : answer with reason

Question 6

A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\displaystyle\lim_{x\to 0} \frac{f(x) - \frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

(a) $$\lim_{n\to\infty}\left|\frac{\frac{(n+2)x^{n+1}}{3^{n+2}}}{\frac{(n+1)x^n}{3^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)}{(n+1)}\frac{x}{3}\right| = \left|\frac{x}{3}\right| < 1$$

At $x = -3$, the series is $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{3}$, which diverges.

At $x = 3$, the series is $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{3}$, which diverges.

Therefore, the interval of convergence is $-3 < x < 3$.

$4: \begin{cases} 1: \text{sets up ratio test} \\ 1: \text{computes limit} \\ 1: \text{conclusion of ratio test} \\ 1: \text{endpoint conclusion} \end{cases}$

(b) $$\lim_{x\to 0}\frac{f(x)-\frac{1}{3}}{x} = \lim_{x\to 0}\left(\frac{2}{3^2} + \frac{3}{3^3}x + \frac{4}{3^4}x^2 + \cdots\right) = \frac{2}{9}$$

1 : answer

(c) $$\int_0^1 f(x)\,dx = \int_0^1\left(\frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots\right)dx$$

$$= \left(\frac{1}{3}x + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^3 + \cdots + \frac{1}{3^{n+1}}x^{n+1} + \cdots\right)\Bigg|_{x=0}^{x=1}$$

$$= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}} + \cdots$$

$3: \begin{cases} 1: \text{antidifferentiation of series} \\ 1: \text{first three terms for definite integral series} \\ 1: \text{general term} \end{cases}$

(d) The series representing $\int_0^1 f(x)\,dx$ is a geometric series.

Therefore, $$\int_0^1 f(x)\,dx = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}.$$

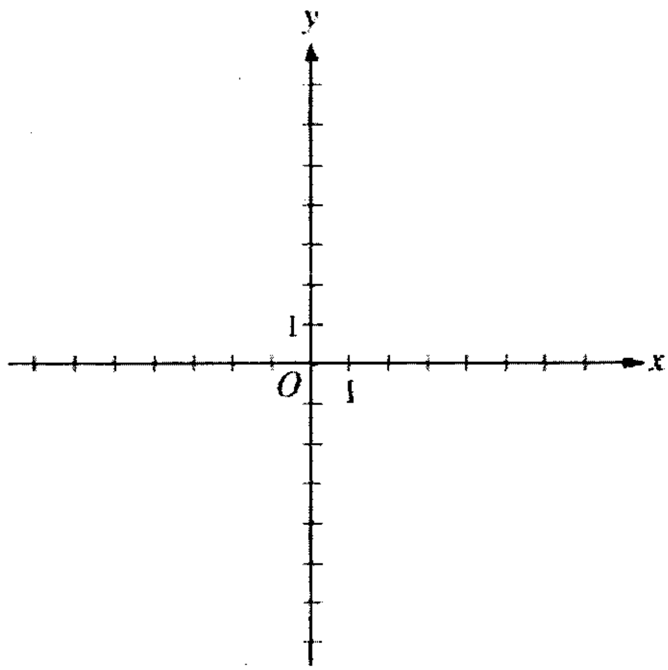
1 : answer

1997 BC1

During the time period from $t = 0$ to $t = 6$ seconds, a particle moves along the path given by $x(t) = 3\cos(\pi t)$ and $y(t) = 5\sin(\pi t)$.

(a) Find the position of the particle when $t = 2.5$.

(b) On the axes provided below, sketch the graph of the path of the particle from $t = 0$ to $t = 6$. Indicate the direction of the particle along its path.



(c) How many times does the particle pass through the point found in part (a)?

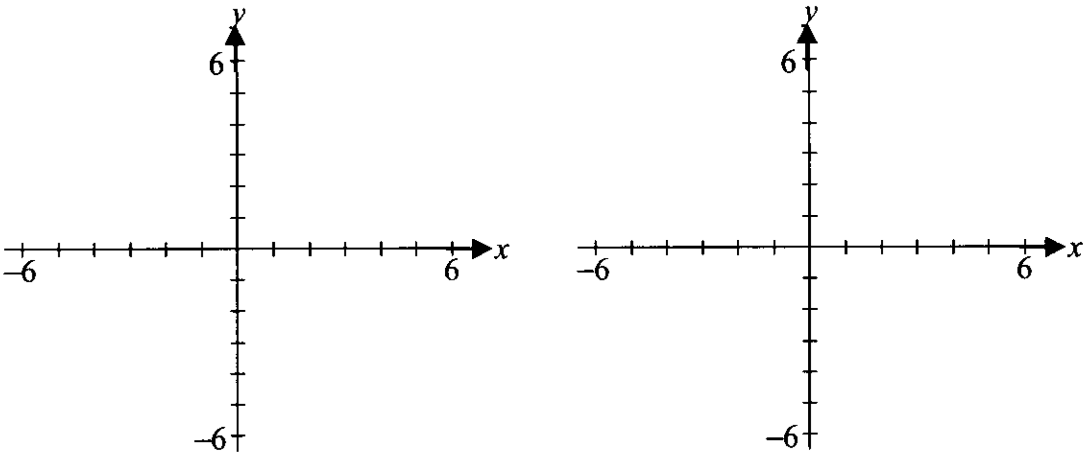
(d) Find the velocity vector for the particle at any time t.

(e) Write and evaluate an integral expression, in terms of sine and cosine, that gives the distance the particle travels from $t = 1.25$ to $t = 1.75$.

1996 AB4/BC4

This problem deals with functions defined by $f(x) = x + b \sin x$, where b is a positive constant and $-2\pi \le x \le 2\pi$.

(a) Sketch the graphs of two of these functions, $y = x + \sin x$ and $y = x + 3\sin x$.



(b) Find the x-coordinates of all points, $-2\pi \le x \le 2\pi$, where the line $y = x + b$ is tangent to the graph of $f(x) = x + b \sin x$.

(c) Are the points of tangency described in part (b) relative maximum points of f? Why?

(d) For all values of $b > 0$, show that all inflection points of the graph of f lie on the line $y = x$.

1995 BC1

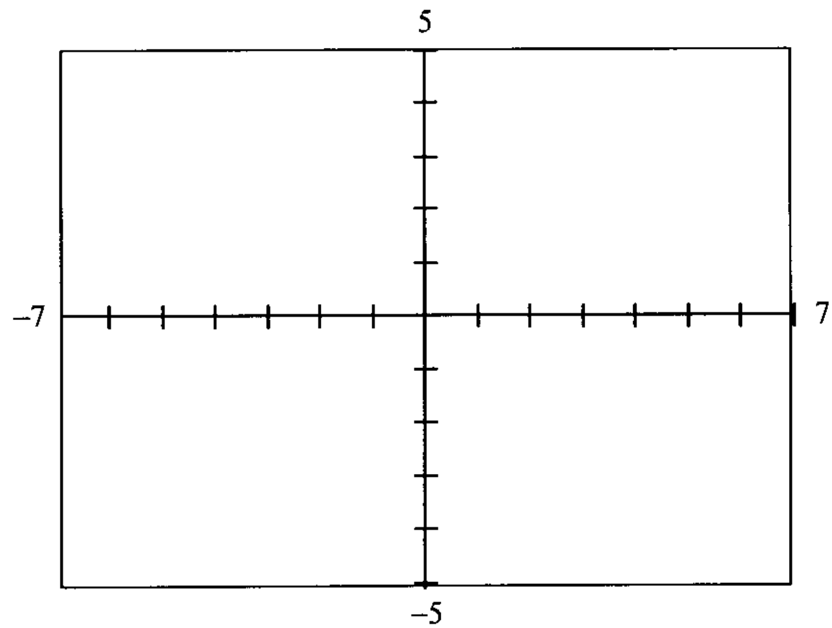
Two particles move in the xy-plane. For time $t \geq 0$, the position of particle A is given by $x = t - 2$ and $y = (t-2)^2$, and the position of particle B is given by $x = \frac{3t}{2} - 4$ and $y = \frac{3t}{2} - 2$.

(a) Find the velocity vector for each particle at time $t = 3$.

(b) Set up an integral expression that gives the distance traveled by particle A from $t = 0$ to $t = 3$. Do not evaluate.

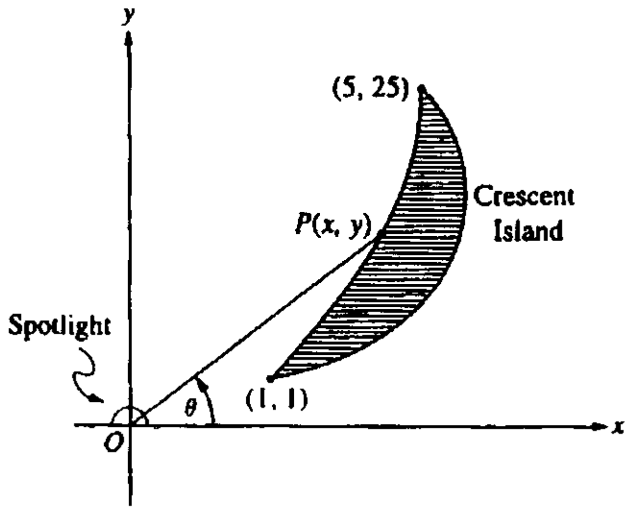
(c) Determine the exact time at which the particles collide; that is, when the particles are at the same point at the same time. Justify your answer.

(d) In the viewing window provided below, sketch the paths of particles A and B from $t = 0$ until they collide. Indicate the direction of each particle along its path.



Viewing Window
$[-7,7] \times [-5,5]$

1996 BC6



Note: **Figure not drawn to scale.**

The figure above shows a spotlight shining on point $P(x, y)$ on the shoreline of Crescent Island. The spotlight is located at the origin and is rotating. The portion of the shoreline on which the spotlight shines is in the shape of the parabola $y = x^2$ from the point $(1,1)$ to the point $(5,25)$. Let θ be the angle between the beam of light and the positive x-axis.

(a) For what values of θ between 0 and 2π does the spotlight shine on the shoreline?

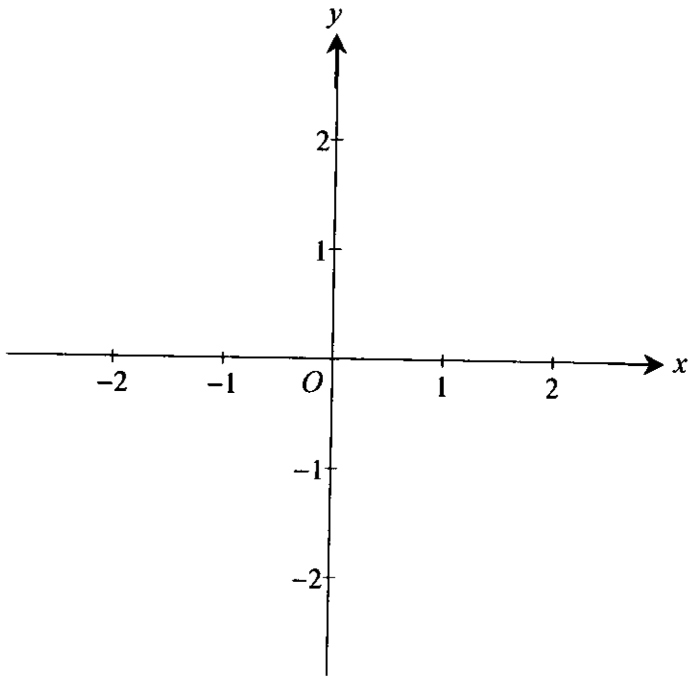
(b) Find the x- and y-coordinates of point P in terms of $\tan\theta$.

(c) If the spotlight is rotating at the rate of one revolution per minute, how fast is the point P traveling along the shoreline at the instant it is at the point $(3,9)$?

1993 BC4

Consider the polar curve $r = 2\sin(3\theta)$ for $0 \le \theta \le \pi$.

(a) In the xy-plane provided below, sketch the curve.



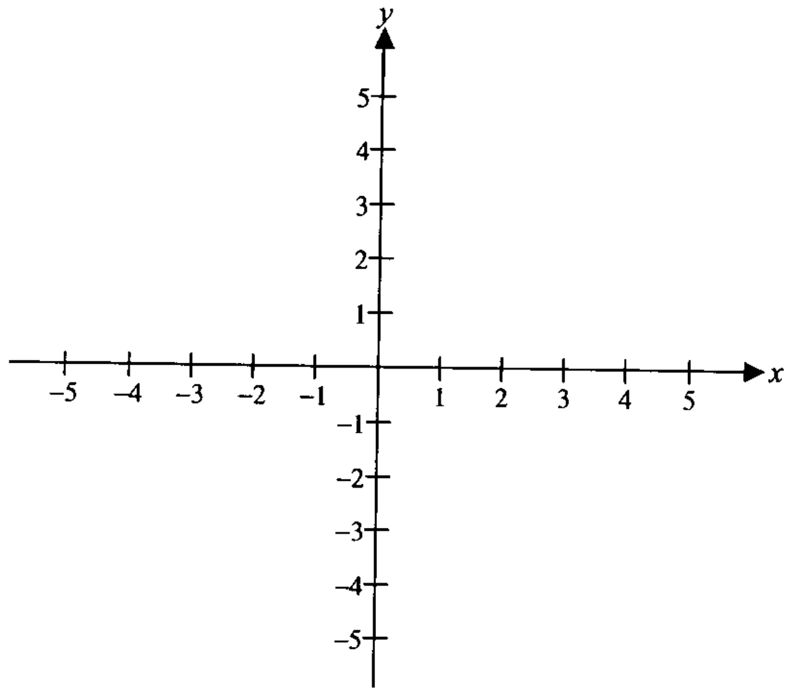
(b) Find the area of the region inside the curve.

(c) Find the slope of the curve at the point where $\theta = \frac{\pi}{4}$.

1990 BC4

Let R be the region inside the graph of the polar curve $r = 2$ and outside the graph of the polar curve $r = 2(1 - \sin\theta)$.

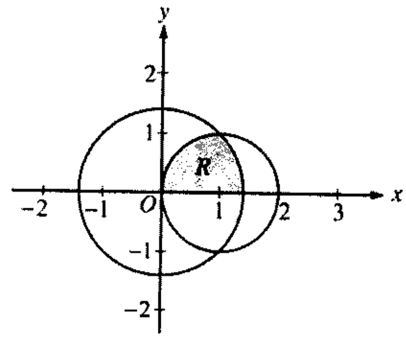
(a) Sketch the two polar curves in the xy-plane provided below and shade the region R
.



(b) Find the area of R .

Question 2

The figure above shows the graphs of the circles $x^2 + y^2 = 2$ and $(x-1)^2 + y^2 = 1$. The graphs intersect at the points $(1,1)$ and $(1,-1)$. Let R be the shaded region in the first quadrant bounded by the two circles and the x-axis.



(a) Set up an expression involving one or more integrals with respect to x that represents the area of R.

(b) Set up an expression involving one or more integrals with respect to y that represents the area of R.

(c) The polar equations of the circles are $r = \sqrt{2}$ and $r = 2\cos\theta$, respectively. Set up an expression involving one or more integrals with respect to the polar angle θ that represents the area of R.

(a) Area $= \int_0^1 \sqrt{1-(x-1)^2}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

OR

Area $= \frac{1}{4}\left(\pi \cdot 1^2\right) + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

$3: \begin{cases} 1: \text{integrand for larger circle} \\ 1: \text{integrand or geometric area for smaller circle} \\ 1: \text{limits on integral(s)} \end{cases}$

Note: < −1 > if no addition of terms

(b) Area $= \int_0^1 \left(\sqrt{2-y^2} - \left(1-\sqrt{1-y^2}\right)\right)dy$

$3: \begin{cases} 1: \text{limits} \\ 2: \text{integrand} \\ \quad < -1 > \text{reversal} \\ \quad < -1 > \text{algebra error in solving for } x \\ \quad < -1 > \text{add rather than subtract} \\ \quad < -2 > \text{other errors} \end{cases}$

(c) Area $= \int_0^{\pi/4} \frac{1}{2}\left(\sqrt{2}\right)^2 d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

OR

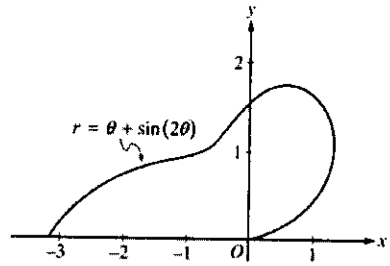
Area $= \frac{1}{8}\pi\left(\sqrt{2}\right)^2 + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

$3: \begin{cases} 1: \text{integrand or geometric area for larger circle} \\ 1: \text{integrand for smaller circle} \\ 1: \text{limits on integral(s)} \end{cases}$

Note: < −1 > if no addition of terms

Question 2

The curve above is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.



(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

(a) Area $= \frac{1}{2}\int_0^{\pi} r^2 \, d\theta$

$= \frac{1}{2}\int_0^{\pi} (\theta + \sin(2\theta))^2 \, d\theta = 4.382$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $-2 = r\cos(\theta) = (\theta + \sin(2\theta))\cos(\theta)$
$\theta = 2.786$

$2 : \begin{cases} 1 : \text{equation} \\ 1 : \text{answer} \end{cases}$

(c) Since $\frac{dr}{d\theta} < 0$ for $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, r is decreasing on this interval. This means the curve is getting closer to the origin.

$2 : \begin{cases} 1 : \text{information about } r \\ 1 : \text{information about the curve} \end{cases}$

(d) The only value in $\left[0, \frac{\pi}{2}\right]$ where $\frac{dr}{d\theta} = 0$ is $\theta = \frac{\pi}{3}$.

θ	r
0	0
$\frac{\pi}{3}$	1.913
$\frac{\pi}{2}$	1.571

The greatest distance occurs when $\theta = \frac{\pi}{3}$.

$2 : \begin{cases} 1 : \theta = \frac{\pi}{3} \text{ or } 1.047 \\ 1 : \text{answer with justification} \end{cases}$

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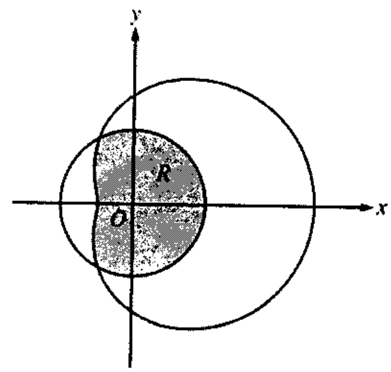
Question 3



The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t),\, y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(a) Area $= \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (3 + 2\cos\theta)^2\, d\theta$
$= 10.370$

$$4 : \begin{cases} 1 : \text{area of circular sector} \\ 2 : \text{integral for section of limaçon} \\ \quad 1 : \text{integrand} \\ \quad 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$$

(b) $\left.\frac{dr}{dt}\right|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = -1.732$

The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.

$$2 : \begin{cases} 1 : \left.\frac{dr}{dt}\right|_{\theta=\pi/3} \\ 1 : \text{interpretation} \end{cases}$$

(c) $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$

$\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} = 0.5$

The particle is moving away from the x-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.

$$3 : \begin{cases} 1 : \text{expression for } y \text{ in terms of } \theta \\ 1 : \left.\frac{dy}{dt}\right|_{\theta=\pi/3} \\ 1 : \text{interpretation} \end{cases}$$